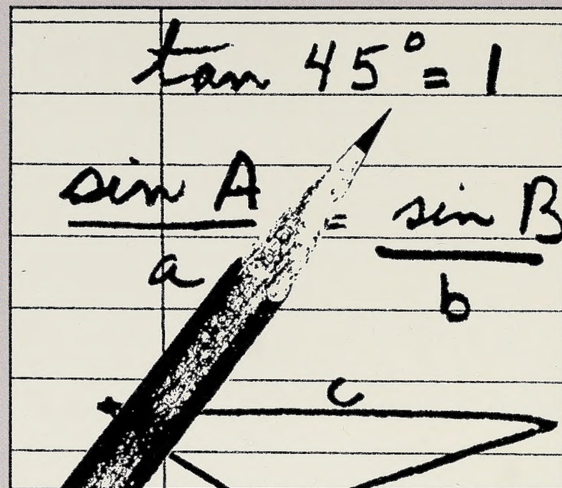
MATHEMATICS 30

Trigonometry



Unit 5

Distance Learning

Alberta EDUCATION

Welcome

Distance Learning

You have chosen an alternate form of learning that allows you to work at your own pace. You will be responsible for your own schedule, for disciplining yourself to study the units thoroughly, and for completing your units regularly. We wish you much success and enjoyment in your studies.

Mathematics 33 Student Module Unit 5 Trigonometry Alberta Distance Learning Centre ISBN No. 0-7741-0186-5

General Information

This information explains the basic layout of each booklet.

- **What You Already Know** and **Review** are to help you look back at what you have previously studied. The questions are to jog your memory and to prepare you for the learning that is going to happen in this unit.

- As you begin each **Topic**, spend a little time looking over the components. Doing this will give you a preview of what will be covered in the topic and will set your mind in the direction of learning.

- **Exploring the Topic** includes the objectives, concept development, and activities for each objective. Use your own papers to arrive at the answers in the activities.

- **Extra Help** reviews the topic. If you had any difficulty with **Exploring the Topic**, you may find this part helpful.

- **Extensions** gives you the opportunity to take the topic one step further.

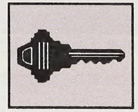
- To summarize what you have learned, and to find instructions on doing the unit assignment, turn to the **Unit Summary** at the end of the unit.

- The **Appendices** include the solutions to **Activities (Appendix A)** and any other charts, tables, etc. which may be referred to in the topics **(Appendix B, etc.)**.

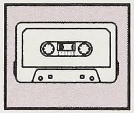
Visual Cues

Visual cues are pictures that are used to identify important areas of the material. They are found throughout the booklet.

An explanation of what they mean is written beside each visual cue.

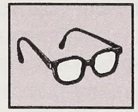


Key Idea
- flagging important ideas

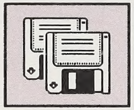


Audiotape
- learning by listening to an audiotape

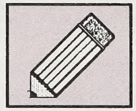
What You Already Know
- reviewing what you already know



Another View
- exploring different perspectives

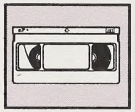


Computer Software
- learning by using computer software



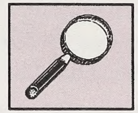
Review
- studying previous concepts

Solutions
- correcting the activities

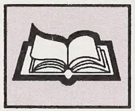


Videotape
- learning by viewing a videotape

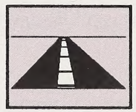
Introduction
- introducing the unit



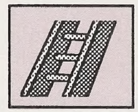
Extra Help
- providing additional study



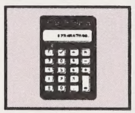
Print Pathway
- choosing a print alternative



What Lies Ahead
- previewing the unit

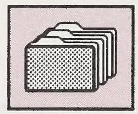


Extensions
- going on with the topic



Calculator
- using your calculator

Exploring the Topic
- actively learning new concepts



What You Have Learned
- summarizing what you have learned

Mathematics 33

Course Overview

Mathematics 33 contains 8 units. Beside each unit is a percentage that indicates what the unit is worth in relation to the rest of the course. The units and their percentages are listed below. You will be studying the unit that is shaded.

Unit 1 Powers and Radicals	10%
Unit 2 Polynomials and Rational Expressions	10%
Unit 3 Functions and Relations	16%
Unit 4 Quadratic Functions and Equations	20%
Unit 5 Trigonometry	16%
Unit 6 Statistics	16%
Unit 7 Annuities	6%
Unit 8 Mortgages and Loans	6%
	100%

Unit Assessment

After completing the unit, you will be given a mark based totally on a unit assignment. This assignment will be found in the Assignment Booklet.

Unit Assignment - 100%

If you are working on a CML terminal, your teacher will determine what this assessment will be. It may be

Unit Assignment - 50%
Supervised Unit Test - 50%

Introduction to Trigonometry

This unit covers topics dealing with trigonometry. Each topic contains explanations, examples, and activities to assist you in understanding trigonometry. If you find you are having difficulty with the explanations and the way the material is presented, there is a section called **Extra Help**. If you would like to extend your knowledge of the topic, there is a section called **Extensions.**

You can evaluate your understanding of each topic by working through the activities. Answers are found in the solutions in **Appendix A**. In several cases there is more than one way to do the question.

Unit 5 Trigonometry

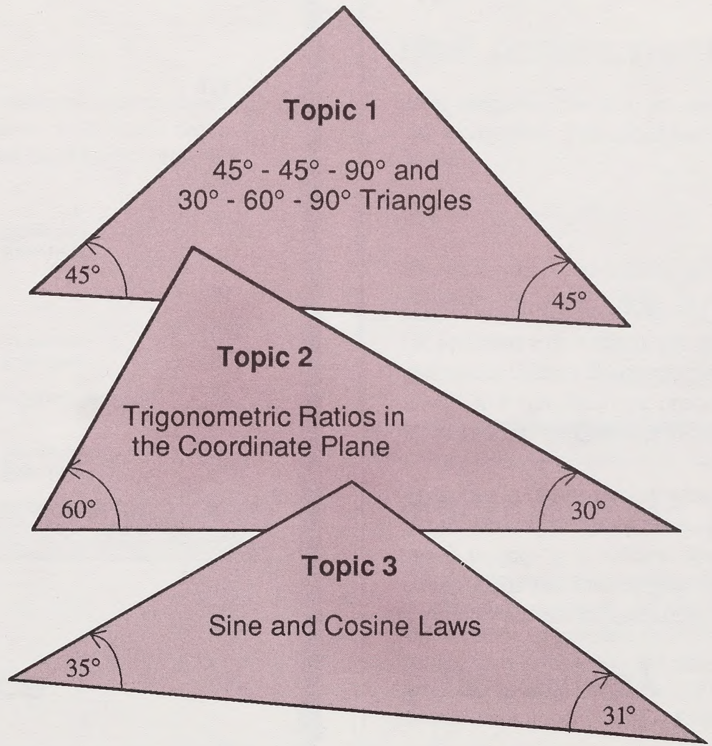
Contents at a Glance

Trigonometry

Triangles are the geometric shapes that are most often used in construction because of their strength. A rectangular box is easily folded. A triangular box is quite difficult to fold. Look at hydro towers, and bridges, and the rafters in your house. Triangles abound everywhere.

Triangles can be difficult to make since any change in one side results in a change in the other sides and a change in the angles. Engineers require precise calculations to produce triangles with the required side lengths and angles. The fundamental skills and concepts that are required will be covered in this unit.

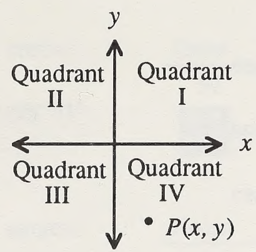
Unit 5 Trigonometry



What You Already Know

Recall the following.

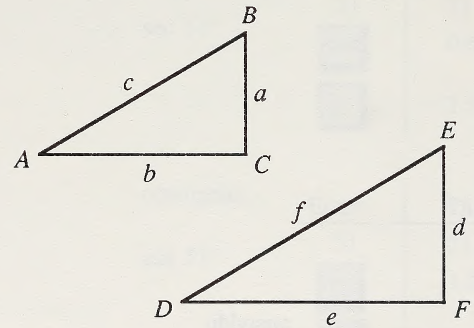
- Cartesian coordinate system:



- Acute angle: angle that measures less than 90°
- Obtuse angle: angle that measures more than 90° but less than 180°
- Right angle: angle that measures 90°
  Symbols are as follows:

- Similar triangles: triangles that have congruent corresponding angles and corresponding sides that are in proportion because they form the same ratio.

For example,



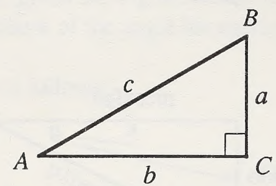
$\angle A = \angle D$

$\angle B = \angle E$

$\angle C = \angle F$

$\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$

- Pythagorean theorem: $c^2 = a^2 + b^2$

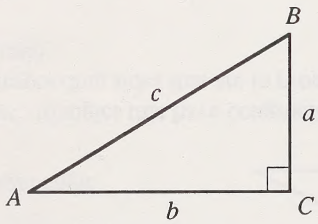


Adaptations are as follows:

$a^2 = c^2 - b^2$

$b^2 = c^2 - a^2$

- Trigonometric functions:



$\sin A = \frac{a}{c}$ $\qquad$ $\csc A = \frac{c}{a}$

$\cos A = \frac{b}{c}$ $\qquad$ $\sec A = \frac{c}{b}$

$\tan A = \frac{a}{b}$ $\qquad$ $\cot A = \frac{b}{a}$

$\sin B = \frac{b}{c}$ $\qquad$ $\csc B = \frac{c}{b}$

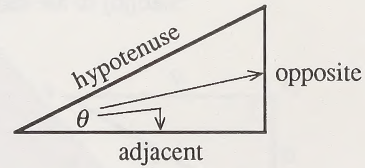
$\cos B = \frac{a}{c}$ $\qquad$ $\sec B = \frac{c}{a}$

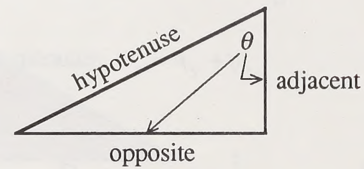
$\tan B = \frac{b}{a}$ $\qquad$ $\cot B = \frac{a}{b}$

$$\sin \theta = \frac{\text{side opposite}}{\text{hypotenuse}} \qquad \csc \theta = \frac{1}{\sin \theta} = \frac{\text{hypotenuse}}{\text{side opposite}}$$

$$\cos \theta = \frac{\text{side adjacent}}{\text{hypotenuse}} \qquad \sec \theta = \frac{1}{\cos \theta} = \frac{\text{hypotenuse}}{\text{side adjacent}}$$

$$\tan \theta = \frac{\text{side opposite}}{\text{side adjacent}} \qquad \cot \theta = \frac{1}{\tan \theta} = \frac{\text{side adjacent}}{\text{side opposite}}$$





- Trigonometric functions on a calculator:

Examples are as follows:

sine:

sin 51°

Enter	Display
51	51
sin	0.777145961

cosine:

cos 51°

Enter	Display
51	51
cos	0.629320391

tangent:

tan 51°

Enter	Display
51	51
tan	1.234897157

cosecant:

csc 51°

Enter	Display
51	51
sin	0.777145961
$1/x$	1.286759566

secant:

sec 51°

Enter	Display
51	51
cos	0.629320391
$1/x$	1.589015729

cotangent:

cot 51°

Enter	Display
51	51
tan	1.234897157
$1/x$	0.809784033

- Determining an angle in degrees given the trigonometric value: In other words, given the trigonometric value, you should be able to find the measure of the angle for each ratio.

Examples are as follows:

$\sin \theta = 0.454$

Enter	Display
0.454	0.454
Inv	0.454
sin	27.00061091

Therefore, $\theta \doteq 27°$.

$\cos\ \theta = 0.574$

Enter	Display
0.574	0.574
Inv	0.574
cos	54.97036838

Therefore, $\theta \doteq 55°$.

$\tan\ \theta = 0.249$

Enter	Display
0.249	0.249
Inv	0.249
tan	13.98230535

Therefore, $\theta \doteq 14°$.

$\cos\ \theta = \frac{3}{4}$

Enter	Display
3	3
÷	3
4	4
=	0.75
Inv	0.75
cos	41.40962211

Therefore, $\theta \doteq 41°$.

- Rationalizing denominators: an operation performed by multiplying by a form of one using the denominator or the irrational part of the denominator divided by itself

For example,

$$\begin{aligned}\frac{20}{\sqrt{2}} &= \frac{20}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{20\sqrt{2}}{\sqrt{2}\sqrt{2}}\\ &= \frac{20\sqrt{2}}{2}\\ &= 10\sqrt{2}\end{aligned}$$

For example,

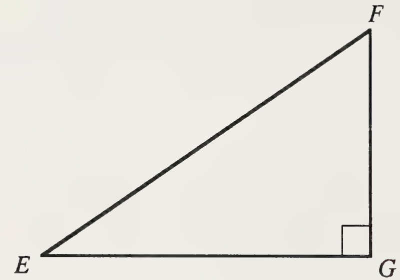
$$\frac{15}{2\sqrt{5}} = \frac{15}{2\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$

$$= \frac{15\sqrt{5}}{(2)(5)}$$

$$= \frac{3\sqrt{5}}{2}$$

Review

Try the following review questions. Questions 1 to 4 refer to the following diagram.



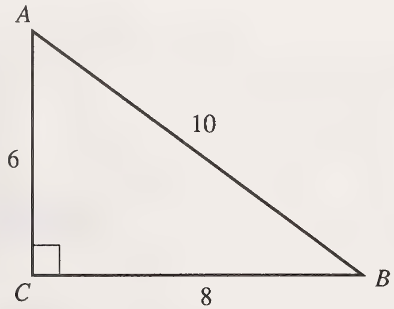
1. Label the sides of triangle *EFG*.

2. Given $e = 10$ cm and $f = 7$ cm, find g.

3. Given $e = 6$ cm and $g = 10$ cm, find f.

4. Given $f = 15$ cm and $g = 27$ cm, find e.

Questions 5 to 8 refer to the following diagram.



5. Write the reduced fraction for each trigonometric ratio.

a. $\sin A$

b. $\sin B$

c. $\csc A$

d. $\csc B$

e. $\cos A$

f. $\cos B$

g. $\sec A$

h. $\sec B$

i. $\tan A$

j. $\tan B$

k. $\cot A$

l. $\cot B$

6. Calculate the measure of angle A using your calculator.

7. Calculate the measure of angle B using your calculator.

8. Why are there no trigonometric functions for angle C?

9. Rationalize the denominator in each following.

a. $\frac{2}{\sqrt{3}}$

b. $\frac{10}{\sqrt{5}}$

c. $\frac{6}{5\sqrt{3}}$

d. $\frac{4}{3\sqrt{2}}$

Now go to the **Review** solutions in **Appendix A**.

Topic 1 45° - 45° - 90° and 30° - 60° - 90° Triangles

Introduction

Right-angled triangles provide an interesting and useful beginning to the study of trigonometry. The exact measures of their sides can be determined because of their unique properties.

In this topic you will learn to determine the exact relative lengths of the sides of these triangles and to calculate the exact values of the trigonometric ratios.

What Lies Ahead

Throughout this topic you will learn to

1. calculate the exact relative measures of the sides of 45° - 45° - 90° triangles
2. calculate the exact relative measures of the sides of 30° - 60° - 90° triangles
3. calculate the exact values of the trigonometric ratios for 45° - 45° - 90° triangles
4. calculate the exact values of the trigonometric ratios for 30° - 60° - 90° triangles

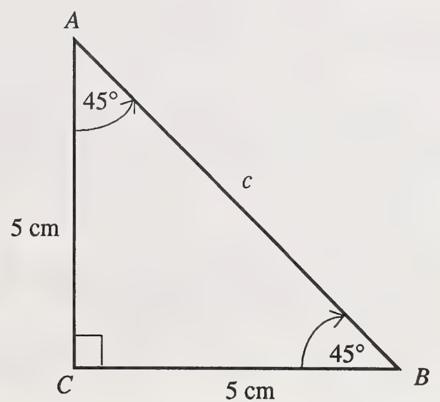
Now that you know what to expect, turn the page to begin your study of 45° - 45° - 90° and 30° - 60° - 90° triangles.

Exploring Topic 1

Activity 1

Calculate the exact relative measures of the sides of 45° - 45° - 90° triangles.

Begin by drawing a 45° - 45° - 90° isosceles triangle with sides of length 5 cm. Label the vertices *ABC*. An isosceles triangle has equal base angles and two sides of equal length.



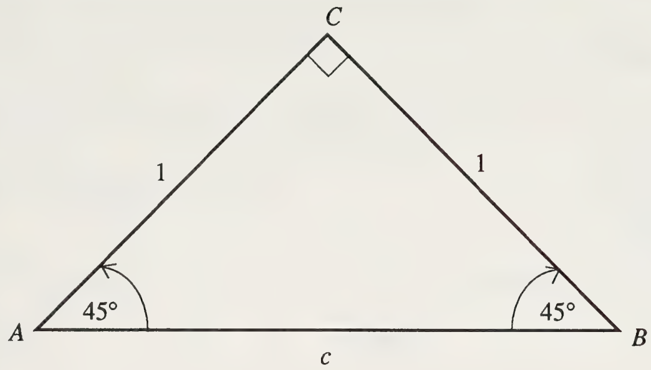
The Pythagorean theorem is used to calculate the length of the hypotenuse.

$$c^2 = a^2 + b^2$$
$$c^2 = (5\text{ cm})^2 + (5\text{ cm})^2$$
$$c^2 = 25\text{ cm}^2 + 25\text{ cm}^2$$
$$c^2 = 50\text{ cm}^2$$
$$c = \sqrt{50\text{ cm}^2}$$
$$c = 5\sqrt{2}\text{ cm}$$

This can be estimated to be $c \doteq 7.07$ cm. Notice the value $c = 5\sqrt{2}$ cm is an exact value. The exact value of the irrational number $\sqrt{2}$ cannot be expressed in a simple form. You will be using these exact irrational numbers throughout this unit. The simplification of $\sqrt{50}$ is shown in the margin box.

$$\begin{aligned}\sqrt{50} &= \sqrt{25 \times 2}\\ &= \sqrt{25}\,\sqrt{2}\\ &= 5\sqrt{2}\end{aligned}$$

Draw a 45° - 45° - 90° triangle with sides of length 1.

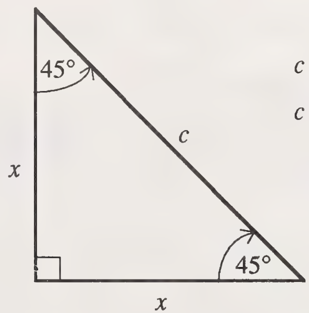


The value of the hypotenuse is calculated as follows:

$$c^2 = a^2 + b^2$$
$$c^2 = (1)^2 + (1)^2$$
$$c^2 = 1 + 1$$
$$c^2 = 2$$
$$c = \sqrt{2}$$

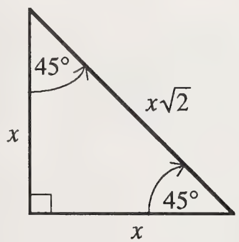
Notice the factor $\sqrt{2}$ has appeared in the value of c in this example as it did when the lengths of the sides were 5.

Do you think all 45° - 45° - 90° triangles share this characteristic? You can try many triangles with various lengths of sides, but it is easiest to use a variable for the length of the sides. Use x.



$$c^2 = x^2 + x^2$$
$$c^2 = 2x^2$$
$$c = \sqrt{2x^2}$$
$$c = x\sqrt{2}$$

The exact relative lengths of the sides of a 45° - 45° - 90° triangle are shown in the following diagram.



Example 1

What is the length of the hypotenuse of a 45° - 45° - 90° triangle which has a side length of 8.75 cm?

Solution:

The hypotenuse of a 45° - 45° - 90° triangle is equal to the length of one side times $\sqrt{2}$.

$c = 8.75\sqrt{2}$ cm (exactly)

$c \doteq 12.37$ cm (approximately)

Example 2

What are the lengths of the sides of a 45° - 45° - 90° triangle whose hypotenuse is equal to $93.25\sqrt{2}$ cm?

Solution:

$c = x\sqrt{2}$; therefore, $x = \dfrac{c}{\sqrt{2}}$

$$x = \frac{93.25\sqrt{2}\text{ cm}}{\sqrt{2}}$$

$$x = 93.25\text{ cm}$$

Two characteristics of a 45° - 45° - 90° triangle are as follows:

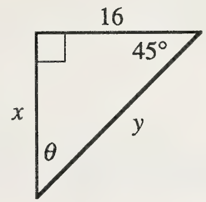
- Both sides are the same length.
- The length of the hypotenuse is $\sqrt{2}$ times the length of a side.

Do the following questions.

1. A 45° - 45° - 90° triangle has one side measuring 30 cm. Find the lengths of the other side and the hypotenuse.

2. An isosceles triangle has one angle measuring 90°. The length of the hypotenuse is 7 m. Find the lengths of the two sides.

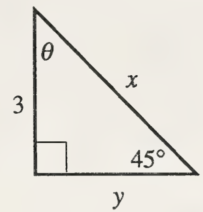
3. Find the missing values in the following.

a. 

$x =$

$y =$

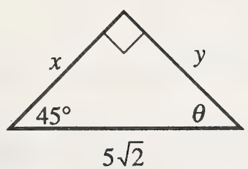
$\theta =$

b. 

$x =$

$y =$

$\theta =$

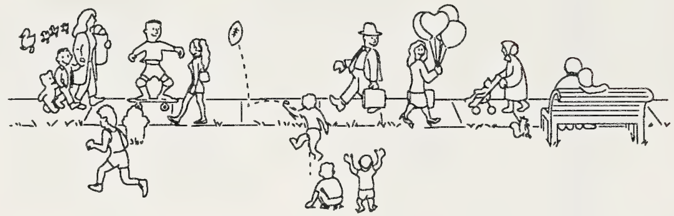
c. 

$x =$

$y =$

$\theta =$

4. A recreational park is in the shape of a square measuring $110\sqrt{2}$ m on each side. A diagonal sidewalk is constructed across this park stretching from one corner to the next. Find the length of the sidewalk.

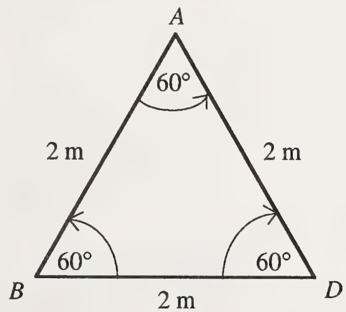


For solutions to **Activity 1**, turn to **Appendix A, Topic 1**.

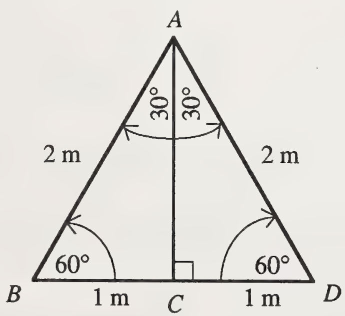
Activity 2

Calculate the exact relative measures of the sides of 30° - 60° - 90° triangles.

Consider an equilateral triangle *ABD* with sides 2 m in length.

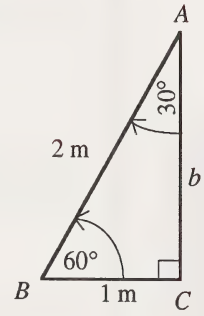


Find the midpoint *C* of the base *BD*. Then, since triangle *ABD* is equilateral, *AC* will bisect angle *A* and will be perpendicular to the base *BD*.



The result is two right-angled triangles.

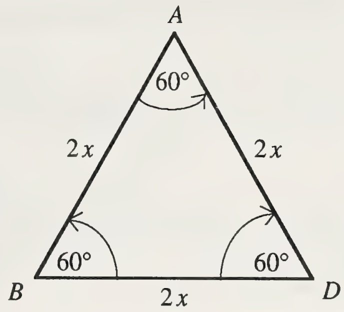
Consider triangle *ABC*.



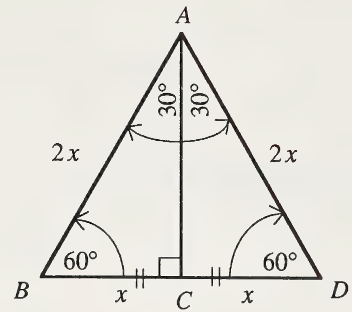
Now calculate the length of side *AC* or *b* by using the Pythagorean formula.

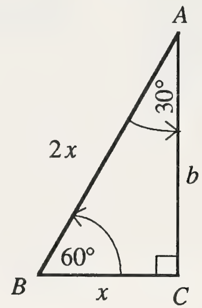
$$c^2 = a^2 + b^2$$
$$b^2 = c^2 - a^2$$
$$b^2 = (2\text{ m})^2 - (1\text{ m})^2$$
$$b^2 = 4\text{ m}^2 - 1\text{ m}^2$$
$$b^2 = 3\text{ m}^2$$
$$b = \sqrt{3\text{ m}^2}$$
$$b = \sqrt{3}\text{ m}$$

Look at the general situation in which the sides of the equilateral triangle measure $2x$ units.



Draw the perpendicular bisector of the base.





The length of side b is calculated as follows:

$$b^2 = c^2 - a^2$$

$$b^2 = (2x)^2 - (x)^2$$

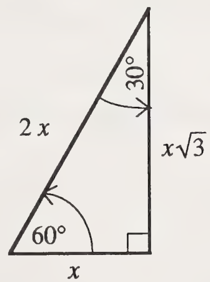
$$b^2 = 4x^2 - x^2$$

$$b^2 = 3x^2$$

$$b = x\sqrt{3}$$

Two characteristics of a 30° - 60° - 90° triangle are as follows:

- The shorter side is one-half the length of the hypotenuse.
- The longer side is $\sqrt{3}$ times the length of the shorter side.

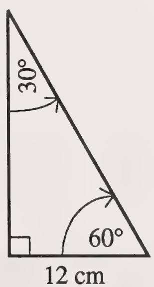


Example 3

A 30° - 60° - 90° triangle has the shortest side measuring 12 cm. Find the lengths of the other two sides.

Solution:

Sketch the triangle.



The hypotenuse is twice the length of the shortest side in a 30° - 60° - 90° triangle. Therefore, the hypotenuse is as follows:

$2(12 \text{ cm}) = 24 \text{ cm}$

The other side is $\sqrt{3}$ times the length of the shortest side. Therefore, the other side is as follows:

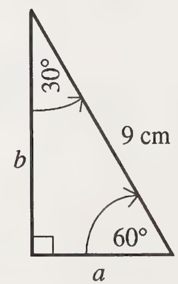
$\sqrt{3}(12 \text{ cm}) = 12\sqrt{3} \text{ cm}$

Example 4

A 30° - 60° - 90° triangle has a hypotenuse of 9 cm. Find the lengths of the other sides.

Solution:

Sketch the triangle.



The shortest side a is one-half the length of the hypotenuse. Therefore,

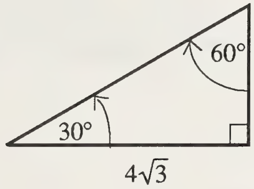
$$\begin{aligned} a &= \frac{1}{2}(9 \text{ cm}) \\ &= 4.5 \text{ cm} \end{aligned}$$

The other side b is $\sqrt{3}$ times the length of the shortest side. Therefore,

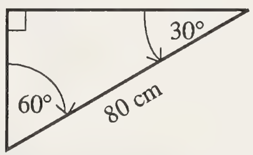
$$\begin{aligned} b &= \sqrt{3}(4.5 \text{ cm}) \\ &= 4.5\sqrt{3} \text{ cm} \end{aligned}$$

Do any four of the following questions.

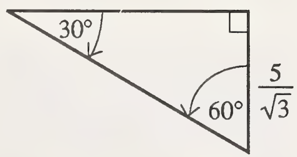
1. A 30° - 60° - 90° triangle is shown. Find the lengths of the other sides.



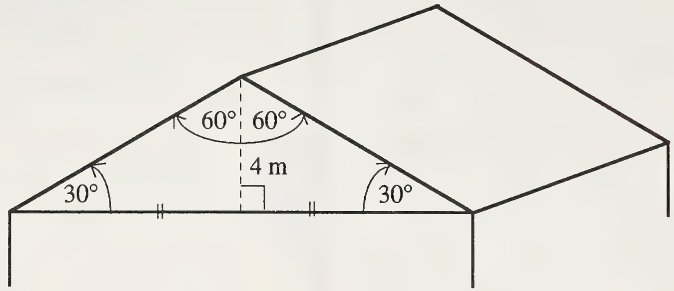
2. A 30° - 60° - 90° triangle is shown. Find the lengths of the other sides.



3. A 30° - 60° - 90° triangle is shown. Find the lengths of the other sides.



4. The shape of a gable of a rectangular house is shown. The house is 15 m long. Find the area of the entire roof.



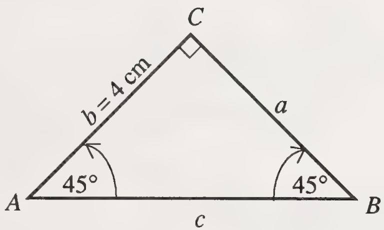
5. Two fire towers are 10 km apart. A fire is spotted by one tower and is measured to be 30° away from the second tower. The second tower measures the angle relative to the first tower to be 90°. How far is the fire from the first tower? How far is the fire from the second tower?

For solutions to **Activity 2**, turn to **Appendix A, Topic 1**.

Activity 3

Calculate the exact values of the trigonometric ratios for 45° - 45° - 90° triangles.

You have already learned to calculate the exact relative measures of the sides of a 45° - 45° - 90° triangle. Now you will use these exact measures to calculate the exact trigonometric ratios. Begin by drawing a 45° - 45° - 90° triangle with one side equal to 4 cm.



The measure of the other side is 4 cm and the measure of the hypotenuse is $4\sqrt{2}$ cm.

Use the trigonometric ratios to calculate the exact trigonometric ratios for this 45° - 45° - 90° triangle.

$$\begin{aligned}\sin A &= \sin 45^\circ \\ &= \frac{\text{side opposite}}{\text{hypotenuse}} \\ &= \frac{a}{c} \\ &= \frac{4\text{ cm}}{4\sqrt{2}\text{ cm}} \\ &= \frac{1}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

$$\begin{aligned}\sin B &= \sin 45^\circ \\ &= \frac{\text{side opposite}}{\text{hypotenuse}} \\ &= \frac{b}{c} \\ &= \frac{4\text{ cm}}{4\sqrt{2}\text{ cm}} \\ &= \frac{1}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

Two characteristics of a 45° - 45° - 90° triangle are as follows:

- The measures of the sides are equal.
- The hypotenuse is equal to $\sqrt{2}$ times the measure of a side.

$$\begin{aligned}\cos A &= \cos 45^\circ\\ &= \frac{\text{side adjacent}}{\text{hypotenuse}}\\ &= \frac{b}{c}\\ &= \frac{4\text{ cm}}{4\sqrt{2}\text{ cm}}\\ &= \frac{1}{\sqrt{2}}\\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

$$\begin{aligned}\cos B &= \cos 45^\circ\\ &= \frac{\text{side adjacent}}{\text{hypotenuse}}\\ &= \frac{a}{c}\\ &= \frac{4\text{ cm}}{4\sqrt{2}\text{ cm}}\\ &= \frac{1}{\sqrt{2}}\\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

$$\begin{aligned}\tan A &= \tan 45^\circ\\ &= \frac{\text{side opposite}}{\text{side adjacent}}\\ &= \frac{a}{b}\\ &= \frac{4\text{ cm}}{4\text{ cm}}\\ &= 1\end{aligned}$$

$$\begin{aligned}\tan B &= \tan 45^\circ\\ &= \frac{\text{side opposite}}{\text{side adjacent}}\\ &= \frac{b}{a}\\ &= \frac{4\text{ cm}}{4\text{ cm}}\\ &= 1\end{aligned}$$

Use your calculator to see if it will produce these answers.

Calculate $\sin 45^\circ$.

Enter	Display
45	45
$\boxed{\sin}$	0.707106781

Is this the same as $\frac{\sqrt{2}}{2}$?

Calculate $\frac{\sqrt{2}}{2}$.

Enter	Display
2	2
$\boxed{\sqrt{\ }}$	1.414213562
$\boxed{\div}$	1.414213562
2	2
$\boxed{=}$	0.707106781

Does this mean that $\frac{\sqrt{2}}{2} = 0.707106781$?

No, $\sqrt{2}$ is an irrational number; that is, it cannot be expressed exactly as a decimal.

You can likewise test $\cos\ 45° = \frac{\sqrt{2}}{2}$ and $\tan\ 45° = 1$.

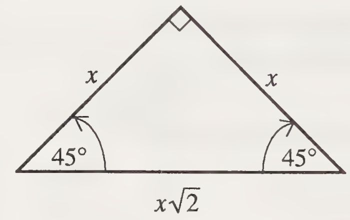
The exact inverse trigonometric ratios are as follows:

$$\begin{aligned}\csc\ 45° &= \frac{2}{\sqrt{2}}\\ &= \frac{2\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}}\\ &= \frac{2\sqrt{2}}{2}\\ &= \sqrt{2}\end{aligned}$$

$$\begin{aligned}\sec\ 45° &= \frac{2}{\sqrt{2}}\\ &= \frac{2\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}}\\ &= \frac{2\sqrt{2}}{2}\\ &= \sqrt{2}\end{aligned}$$

$$\cot\ 45° = 1$$

Use the following triangle to determine the exact trigonometric ratios for 45°.



$$\begin{aligned}\sin\ 45° &= \frac{x}{x\sqrt{2}}\\ &= \frac{1}{\sqrt{2}}\\ &= \frac{1\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}}\\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

$$\begin{aligned}\cos\ 45° &= \frac{x}{x\sqrt{2}}\\ &= \frac{1}{\sqrt{2}}\\ &= \frac{1\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}}\\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

$$\begin{aligned}\tan\ 45° &= \frac{x}{x}\\ &= 1\end{aligned}$$

$$\begin{aligned}\csc\ 45° &= \frac{x\sqrt{2}}{x}\\ &= \sqrt{2}\end{aligned}$$

$$\begin{aligned}\sec\ 45° &= \frac{x\sqrt{2}}{x}\\ &= \sqrt{2}\end{aligned}$$

$$\cot\ 45° = 1$$

If your calculator does not have csc, sec, and cot functions, calculate the value of sin, cos, and tan respectively and take the reciprocal of the values.
For example,

$$\csc\ 45° = \frac{1}{\sin\ 45°}$$

Exact trigonometric ratios are as follows:

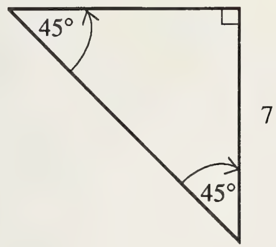
$$\sin\ 45° = \frac{\sqrt{2}}{2} \qquad \csc\ 45° = \sqrt{2}$$

$$\cos\ 45° = \frac{\sqrt{2}}{2} \qquad \sec\ 45° = \sqrt{2}$$

$$\tan\ 45° = 1 \qquad \cot\ 45° = 1$$

Try the following questions.

Determine the exact trigonometric ratios for 45° using the diagram shown.



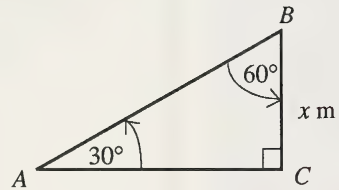
1. sin 45°
2. csc 45°
3. cos 45°
4. sec 45°
5. tan 45°
6. cot 45°

For solutions to **Activity 3**, turn to **Appendix A, Topic 1**.

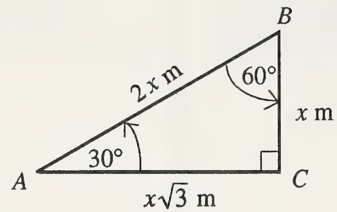
Activity 4

Calculate the exact values of the trigonometric ratios for 30° - 60° - 90° triangles.

Consider the 30° - 60° - 90° triangle *ABC*, with the shortest side equal to x m.



The other side is calculated to be $x \bullet \sqrt{3}$ m. The hypotenuse is calculated to be $2x$ m.



The exact trigonometric values are calculated as follows:

- $\sin A = \sin 30°$

$$\begin{aligned} &= \frac{\text{side opposite}}{\text{hypotenuse}} \\ &= \frac{x \text{ m}}{2x \text{ m}} \\ &= \frac{1}{2} \end{aligned}$$

- $\sin B = \sin 60°$

$$\begin{aligned} &= \frac{\text{side opposite}}{\text{hypotenuse}} \\ &= \frac{x\sqrt{3} \text{ m}}{2x \text{ m}} \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

- $\cos A = \cos 30°$

$$\begin{aligned} &= \frac{\text{side adjacent}}{\text{hypotenuse}} \\ &= \frac{x\sqrt{3} \text{ m}}{2x \text{ m}} \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

- $\cos B = \cos 60°$

$$\begin{aligned} &= \frac{\text{side adjacent}}{\text{hypotenuse}} \\ &= \frac{x \text{ m}}{2x \text{ m}} \\ &= \frac{1}{2} \end{aligned}$$

- $\tan A = \tan 30°$

$$\begin{aligned} &= \frac{\text{side opposite}}{\text{side adjacent}} \\ &= \frac{x \text{ m}}{x\sqrt{3} \text{ m}} \\ &= \frac{1}{\sqrt{3}} \\ &= \frac{\sqrt{3}}{3} \end{aligned}$$

- $\tan B = \tan 60°$

$$\begin{aligned} &= \frac{\text{side opposite}}{\text{side adjacent}} \\ &= \frac{x\sqrt{3} \text{ m}}{x \text{ m}} \\ &= \sqrt{3} \end{aligned}$$

The exact inverse trigonometric ratios are calculated as follows:

- $\csc A = \csc 30°$
$$\begin{aligned} &= \frac{\text{hypotenuse}}{\text{side opposite}} \\ &= \frac{2x \text{ m}}{x \text{ m}} \\ &= 2 \end{aligned}$$

- $\csc B = \csc 60°$
$$\begin{aligned} &= \frac{\text{hypotenuse}}{\text{side opposite}} \\ &= \frac{2x \text{ m}}{x\sqrt{3} \text{ m}} \\ &= \frac{2}{\sqrt{3}} \\ &= \frac{2\sqrt{3}}{3} \end{aligned}$$

- $\sec A = \sec 30°$
$$\begin{aligned} &= \frac{\text{hypotenuse}}{\text{side adjacent}} \\ &= \frac{2x \text{ m}}{x\sqrt{3} \text{ m}} \\ &= \frac{2}{\sqrt{3}} \\ &= \frac{2\sqrt{3}}{3} \end{aligned}$$

- $\sec B = \sec 60°$
$$\begin{aligned} &= \frac{\text{hypotenuse}}{\text{side adjacent}} \\ &= \frac{2x \text{ m}}{x \text{ m}} \\ &= 2 \end{aligned}$$

- $\cot A = \cot 30°$
$$\begin{aligned} &= \frac{\text{side adjacent}}{\text{side opposite}} \\ &= \frac{x\sqrt{3} \text{ m}}{x \text{ m}} \\ &= \sqrt{3} \end{aligned}$$

- $\cot B = \cot 60°$
$$\begin{aligned} &= \frac{\text{side adjacent}}{\text{side opposite}} \\ &= \frac{x \text{ m}}{x\sqrt{3} \text{ m}} \\ &= \frac{1}{\sqrt{3}} \\ &= \frac{\sqrt{3}}{3} \end{aligned}$$

Exact trigonometric ratios are as follows:

$\sin 30° = \frac{1}{2}$ $\qquad$ $\sin 60° = \frac{\sqrt{3}}{2}$

$\cos 30° = \frac{\sqrt{3}}{2}$ $\qquad$ $\cos 60° = \frac{1}{2}$

$\tan 30° = \frac{\sqrt{3}}{3}$ $\qquad$ $\tan 60° = \sqrt{3}$

$\csc 30° = 2$ $\qquad$ $\csc 60° = \frac{2\sqrt{3}}{3}$

$\sec 30° = \frac{2\sqrt{3}}{3}$ $\qquad$ $\sec 60° = 2$

$\cot 30° = \sqrt{3}$ $\qquad$ $\cot 60° = \frac{\sqrt{3}}{3}$

Use your calculator as a test to see if it will produce these values (or approximations of these exact values).

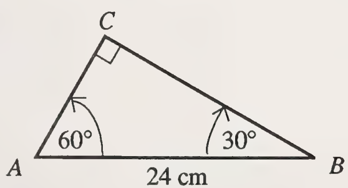
Enter	Display
30	30
$\boxed{\text{sin}}$	0.5

The answer 0.5 is the same as $\frac{1}{2}$.

Try at least two of the following three questions.

1. Verify all six trigonometric ratios for each of 30° and 60° using your calculator.

2. Use the following triangle to calculate the twelve exact trigonometric ratios for 30° and 60°.



3. Fill in the blanks with the exact value. (Do not refer to the notes.)

a. $\cos 60° =$ _______ b. $\sin 30° =$ _______

c. $\tan 60° =$ _______ d. $\tan$ _______$° = \frac{\sqrt{3}}{3}$

e. $\csc$ _______$° = \frac{2\sqrt{3}}{3}$ f. $\sec$ _______$° = 2$

g. $\cos 30° =$ _______ h. $\sin$ _______$° = \frac{\sqrt{3}}{2}$

i. $\cot$ _______$° = \frac{\sqrt{3}}{3}$ j. $\cot 30° =$ _______

k. $\sec 30° =$ _______ l. $\csc 30° =$ _______

For solutions to **Activity 4**, turn to **Appendix A, Topic 1**.

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section.

} You may decide to do both.

Extra Help

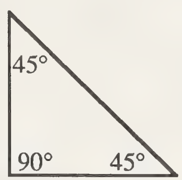
You may study this section by doing **Part A** or **Part B** or both. **Part A** consists of an audiotape while **Part B** is the print section. Whichever way you choose to study this section, complete the exercise at the end of the print section.

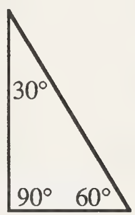
Part A

Audio Activity

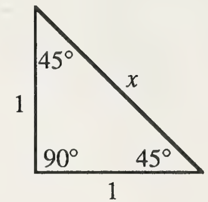
Insert the audiotape titled **Mathematics 33 – Special Triangles** into your tape recorder and follow the instructions on the tape.

1 Special Triangles





2 45° - 45° - 90° Triangle

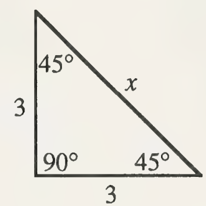


$x^2 = 1^2 + 1^2$ (Pythagorean theorem)

$x^2 = 2$

$x = \sqrt{2}$

3



$x^2 = 3^2 + 3^2$

$x^2 = 9 + 9$

$x^2 = 18$

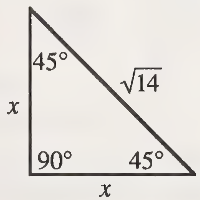
$x = \sqrt{18}$

$x = 3\sqrt{2}$

4 The relationship of the lengths is 3 to 3 to $3\sqrt{2}$.

This relationship reduces to 1 to 1 to $\sqrt{2}$.

5



$$x^2 + x^2 = \left(\sqrt{14}\right)^2$$

Solve for x.

6 $\sqrt{7}$ to $\sqrt{7}$ to $\sqrt{14}$.

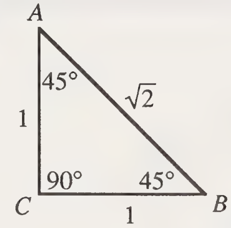
Dividing each length by $\sqrt{7}$ gives the following:

$\frac{\sqrt{7}}{\sqrt{7}}$ to $\frac{\sqrt{7}}{\sqrt{7}}$ to $\frac{\sqrt{14}}{\sqrt{7}}$ which gives

1 to 1 to $\sqrt{\frac{14}{7}}$ or

1 to 1 to $\sqrt{2}$

7



$$\sin A = \frac{\text{opp}}{\text{hyp}} = \frac{1}{\sqrt{2}} = \frac{1 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos A = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{2}} = \frac{1 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan A = \frac{\text{opp}}{\text{adj}} = \frac{1}{1} = 1$$

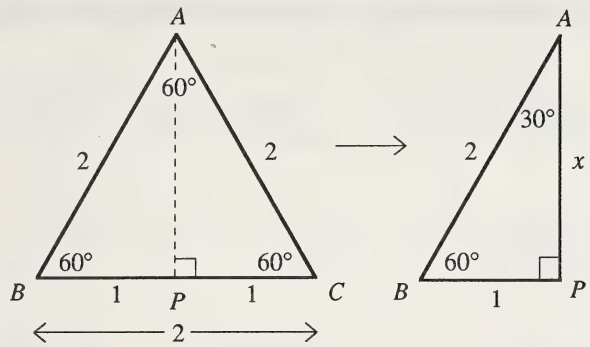
Also,

$$\sin B = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos B = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan B = \frac{1}{1} = 1$$

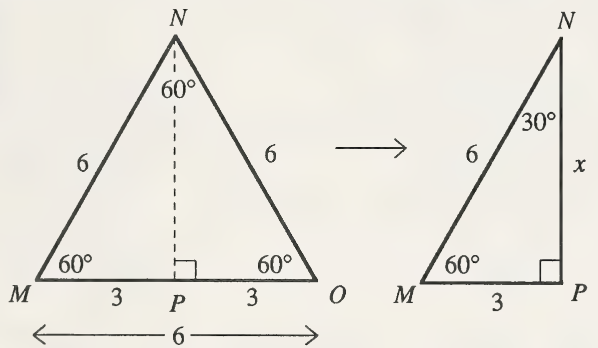
8



9

$$2^2 = 1^2 + x^2$$
$$x^2 = 2^2 - 1^2$$
$$x^2 = 4 - 1$$
$$x^2 = 3$$

10

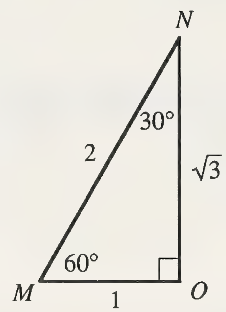


11

$$6^2 = 3^2 + x^2$$
$$x^2 =$$
Solve for x.

12 The relationship 3 to $3\sqrt{3}$ to 6 is the same as 1 to $\sqrt{3}$ to 2.

13



$$\sin 30° = \frac{\text{opp}}{\text{hyp}} = \frac{1}{2} \qquad \sin 60° = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}}{2}$$

$$\cos 30° = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{3}}{2} \qquad \cos 60° = \frac{\text{adj}}{\text{hyp}} = \frac{1}{2}$$

$$\tan 30° = \frac{\text{opp}}{\text{adj}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \tan 60° = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

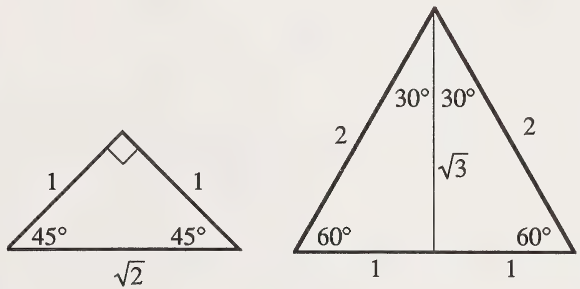
14 Memorize.

$\sin 30° = \frac{1}{2}$ $\quad$ $\sin 45° = \frac{\sqrt{2}}{2}$ $\quad$ $\sin 60° = \frac{\sqrt{3}}{2}$

$\cos 30° = \frac{\sqrt{3}}{2}$ $\quad$ $\cos 45° = \frac{\sqrt{2}}{2}$ $\quad$ $\cos 60° = \frac{1}{2}$

$\tan 30° = \frac{\sqrt{3}}{3}$ $\quad$ $\tan 45° = 1$ $\quad$ $\tan 60° = \frac{\sqrt{3}}{1}$

15



S sine
O opposite
H hypotenuse

$\text{sine} = \frac{\text{opposite}}{\text{hypotenuse}}$ $\quad$ $\text{cosecant} = \frac{\text{hypotenuse}}{\text{opposite}}$

C cosine
A adjacent
H hypotenuse

$\text{cosine} = \frac{\text{adjacent}}{\text{hypotenuse}}$ $\quad$ $\text{secant} = \frac{\text{hypotenuse}}{\text{adjacent}}$

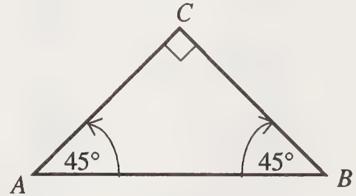
T tangent
O opposite
A adjacent

$\text{tangent} = \frac{\text{opposite}}{\text{adjacent}}$ $\quad$ $\text{cotangent} = \frac{\text{adjacent}}{\text{opposite}}$

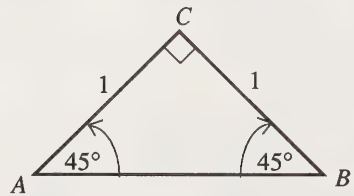
Part B

The exact trigonometric ratios for 30°, 45°, and 60° are determined from two kinds of triangles: 45° - 45° - 90° and 30° - 60° - 90°. You can always generate these triangles and the trigonometric values when you are doing a test or if your memory fails.

For example, sketch a 45° - 45° - 90° triangle. It does not have to be drawn to scale. Label the angles and vertices.



Assign a value of 1 to the shorter sides of the triangle.



Using the Pythagorean theorem, calculate the hypotenuse.

$$c^2 = a^2 + b^2$$
$$c^2 = 1 + 1$$
$$c^2 = 2$$
$$c = \sqrt{2}$$

Calculate the exact values for the six trigonometric functions.

$$\sin 45° = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\csc 45° = \frac{\text{hypotenuse}}{\text{side opposite}} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

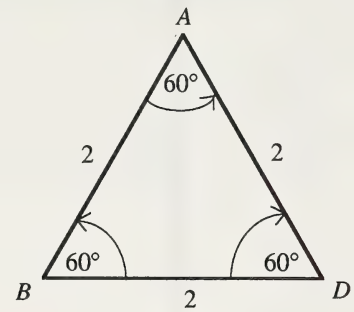
$$\cos 45° = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\sec 45° = \frac{\text{hypotenuse}}{\text{side adjacent}} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

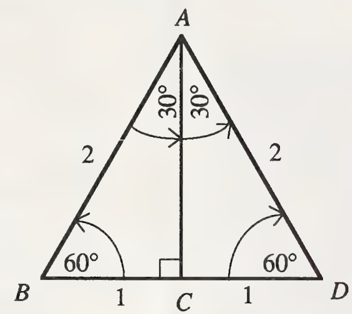
$$\tan 45° = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{1}{1} = 1$$

$$\cot 45° = \frac{\text{side adjacent}}{\text{side opposite}} = \frac{1}{1} = 1$$

For a 30° - 60° - 90° triangle, sketch an equilateral triangle, assign a length of 2 to the sides, and label the triangle.



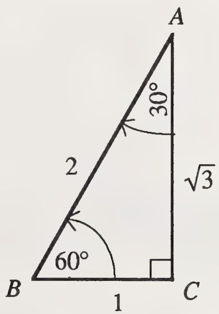
Sketch the segment joining vertex *A* to the midpoint of the base *BD*. Then *AC* will bisect angle *A* and will be perpendicular to the base *BD*.



Now two 30° - 60° - 90° triangles are present with the length of the shorter side equal to 1 and the length of the hypotenuse equal to 2.

Calculate the length of the other side using the Pythagorean theorem.

$$c^2 = a^2 + b^2$$
$$b^2 = c^2 - a^2$$
$$b^2 = 2^2 - 1^2$$
$$b^2 = 4 - 1$$
$$b^2 = 3$$
$$b = \sqrt{3}$$



Calculate the exact values for the trigonometric functions.

$$\sin 30° = \frac{1}{2}$$
$$\cos 30° = \frac{\sqrt{3}}{2}$$
$$\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$
$$\csc 30° = 2$$
$$\sec 30° = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$
$$\cot 30° = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\sin 60° = \frac{\sqrt{3}}{2}$$
$$\cos 60° = \frac{1}{2}$$
$$\tan 60° = \sqrt{3}$$
$$\csc 60° = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$
$$\sec 60° = 2$$
$$\cot 60° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Always remember to rationalize a radical denominator.

$$\frac{3}{\sqrt{3}} = \frac{3}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{3}}{3} = \sqrt{3}$$

Now try it yourself. With only a blank piece of paper derive the exact values of the trigonometric functions for 30°, 45°, and 60°. This procedure should be memorized since it is much easier than memorizing all of the values. If you do not get it right the first time, try again after referring to the **Extra Help** notes. Also practise this procedure during the rest of this unit whenever you need the values of these ratios.

Do you have trouble remembering the trigonometric ratios? Here is a memory device to help you remember the trigonometric ratios.

SOH CAH TOA (pronounced "soak a toa")

S – sine
O – opposite
H – hypotenuse

$$\sin = \frac{\text{opposite}}{\text{hypotenuse}}$$

C – cosine
A – adjacent
H – hypotenuse

$$\cos = \frac{\text{adjacent}}{\text{hypotenuse}}$$

T – tangent
O – opposite
A – adjacent

$$\tan = \frac{\text{opposite}}{\text{adjacent}}$$

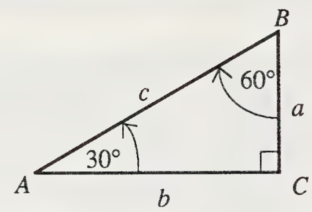
The inverse trigonometric ratios are easy to get from this chart. Just take the reciprocals of sin, cos and tan to get csc, sec, and cot, respectively.

$$\csc = \frac{\text{hypotenuse}}{\text{opposite}} \qquad \sec = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\cot = \frac{\text{adjacent}}{\text{opposite}}$$

Try the following question.

Refer to ΔABC and fill in the blanks.



$\sin\ A = \underline{\ \frac{a}{c}\ }$ $\sin\ B =$ ________

$\csc\ A =$ ________ $\csc\ B =$ ________

$\cos\ A =$ ________ $\cos\ B =$ ________

$\sec\ A =$ ________ $\sec\ B =$ ________

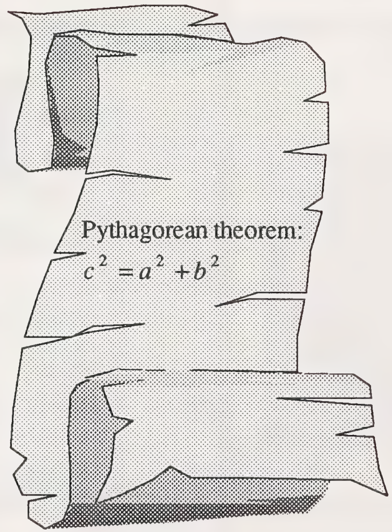
$\tan\ A =$ ________ $\tan\ B =$ ________

$\cot\ A =$ ________ $\cot\ B =$ ________

For solutions to **Extra Help**, turn to **Appendix A, Topic 1**.

Extensions

The Pythagorean theorem was used to calculate the length of any one side of a right triangle. How much do you know about Pythagoras the Greek mathematician? Did Pythagoras really formulate the Pythagorean theorem? Pythagoras was a Greek philosopher and mathematician. All of his writings were destroyed. He founded a brotherhood in the city of Crotona, Italy. Most of its members were killed later in a political uprising. Pythagoras may have been one of those killed. If you are interested in finding out more about Pythagoras and the Pythagorean theorem, you can go to the library and do some research work.



Try to answer the following questions.

1. Who discovered the magic 3-4-5 triangle?

2. Did Pythagoras really formulate the Pythagorean theorem?

3. What is Pythagoreanism?

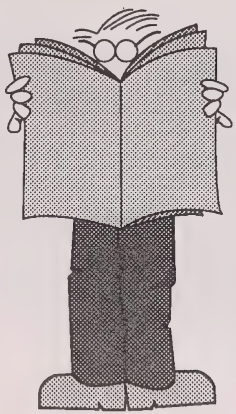
For solutions to **Extensions**, turn to **Appendix A, Topic 1**.

Topic 2 Trigonometric Ratios in the Coordinate Plane

Introduction

The coordinate plane is the tool which makes the concept of trigonometry easier to visualize and master.

In this topic you will learn to appreciate the appropriateness and relative simplicity of trigonometry.



What Lies Ahead

Throughout this topic you will learn to

1. recognize and sketch angles with positive and negative measures in standard position on a coordinate plane
2. express the trigonometric ratios and calculate the reference angle for an angle drawn in standard position on a coordinate plane
3. calculate the trigonometric ratios for any angle
4. determine any two values of x, y, r, and θ given the other two

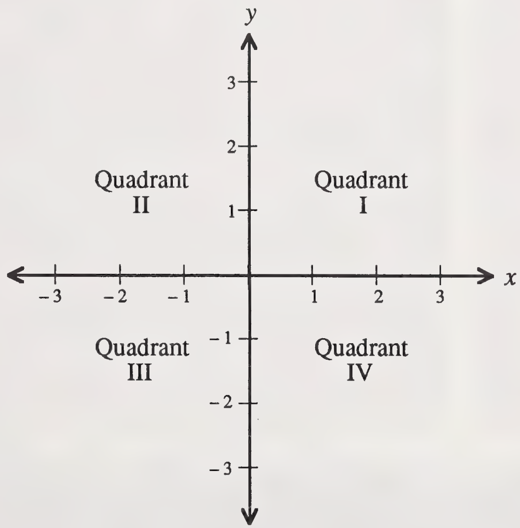
Now that you know what to expect, turn the page to begin your study of trigonometric ratios in the coordinate plane.

Exploring Topic 2

Activity 1

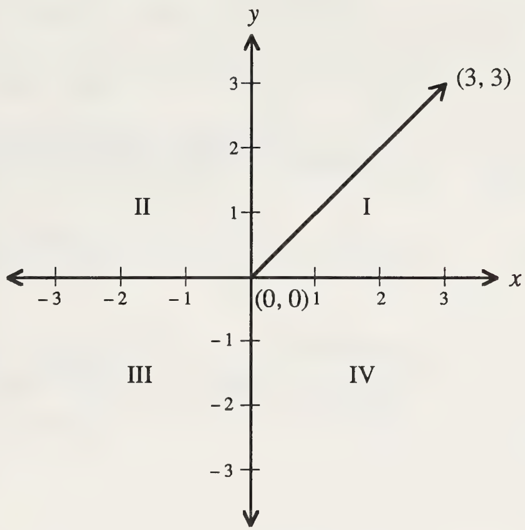
Recognize and sketch angles with positive and negative measures in standard position on a coordinate plane.

The coordinate plane is a two-dimensional surface or grid divided into four regions by two number lines. The number lines are perpendicular to each other and intersect at the origin, the point for which the coordinates are (0, 0). The number lines are called the x-axis and the y-axis. The four regions are called quadrants and are labelled I, II, III, and IV. A plane that shows all of this information follows.



I, II, III, and IV are the Roman numerals for 1, 2, 3, and 4.

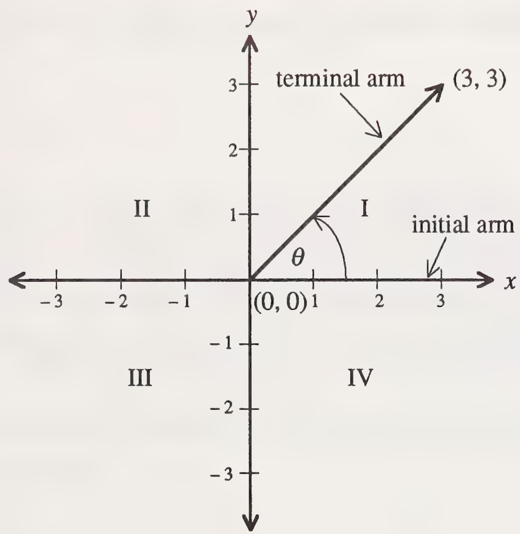
A point (3, 3) is arbitrarily selected in the first quadrant. A ray is drawn from (0, 0) to this point.



This ray is called the **terminal arm**. The positive half of the x-axis is called the **initial arm**.

There is an angle formed between the positive part of the x-axis or the initial arm and the terminal arm. Often an angle is imagined to be formed when the terminal arm starts on the x-axis and rotates counterclockwise to its final position. This angle is shown in the following diagram.

Note: In the ordered pair (3, 3), the first coordinate refers to the x-value and the second coordinate refers to the y-value.

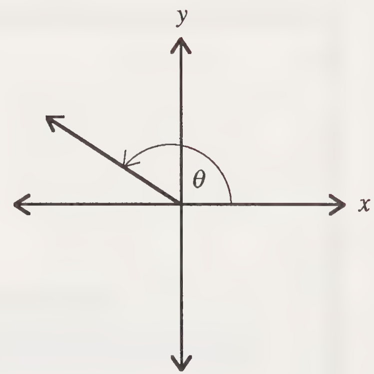


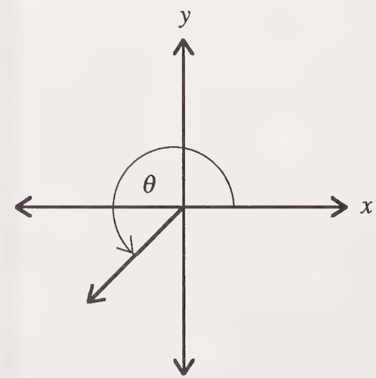
Notice the angle θ is indicated by the curved arrow which starts at the x-axis and ends at the terminal arm. The positive half of the x-axis is called the initial arm since this is where the angle appears to start.

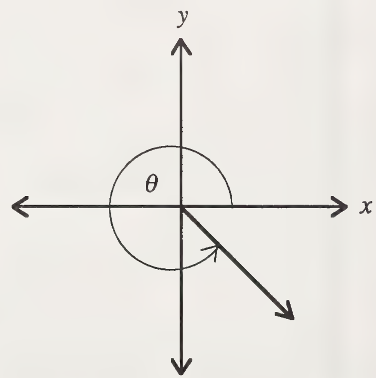
An angle formed by the initial arm and the terminal arm with the vertex at (0, 0) is called an angle in **standard position**.

When the angle is measured from the initial arm to the terminal arm in a counterclockwise direction, the angle is **positive**.

The following are examples of positive angles in standard position.





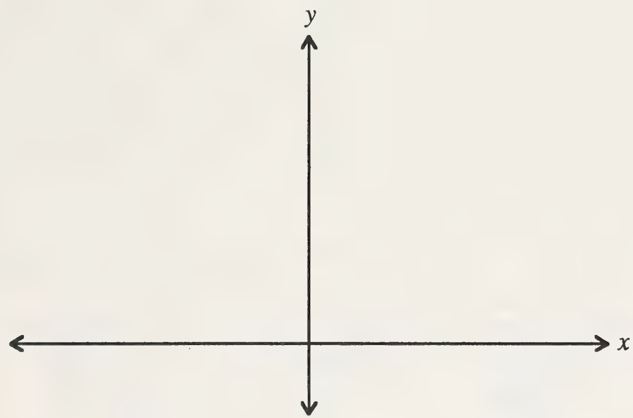


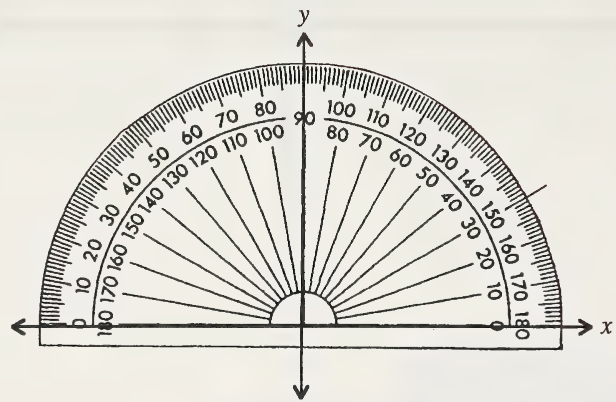
Example 1

Draw a 30° angle in standard position on the coordinate plane.

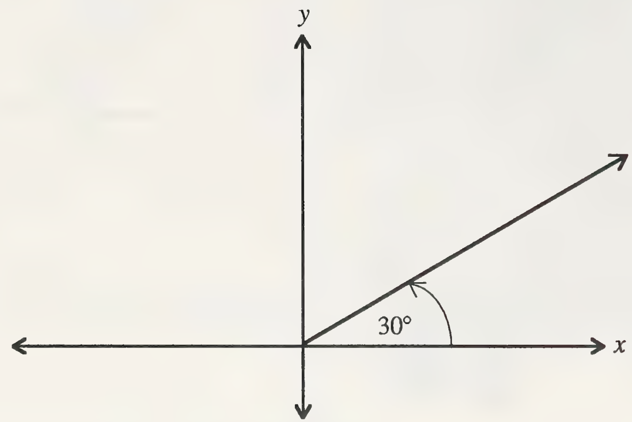
Solution:

Sketch the x- and y-axes. Then place your protractor on the x-axis centred at (0, 0) and place a mark at 30° on your paper.





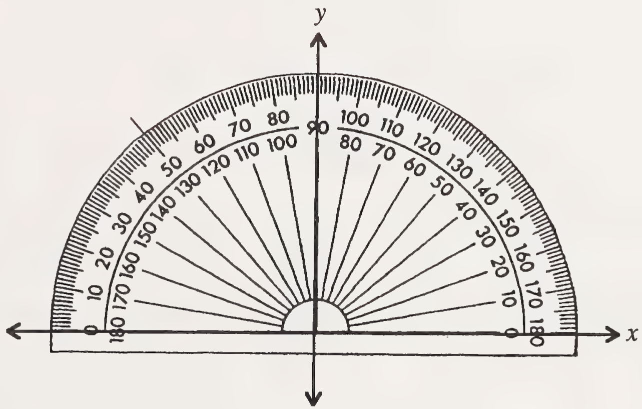
Remove the protractor and draw a ray from the origin through the 30° mark. Label the angle.

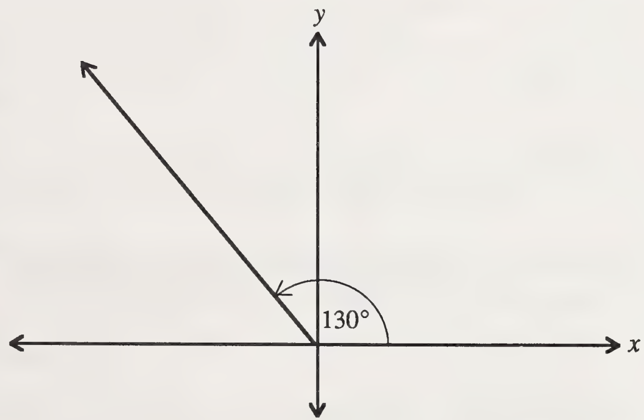


Example 2

Draw a 130° angle in standard position.

Solution:



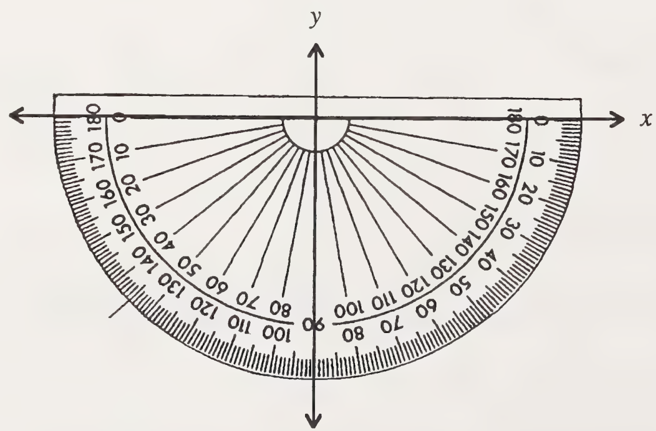


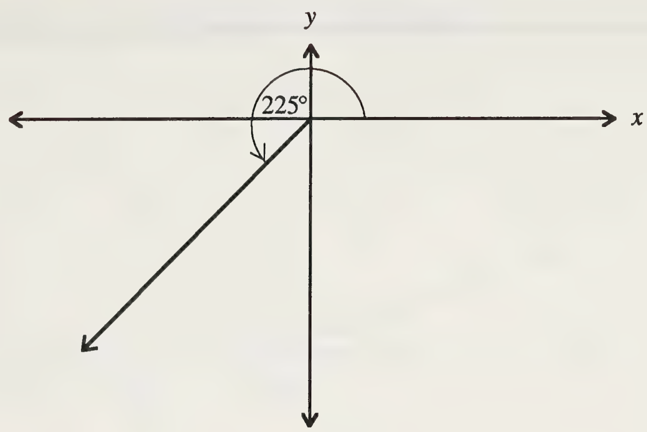
Example 3

Draw a 225° angle in standard position.

Solution:

Subtract 180° from the required angle and measure the resulting angle counterclockwise from the negative *x*-axis.



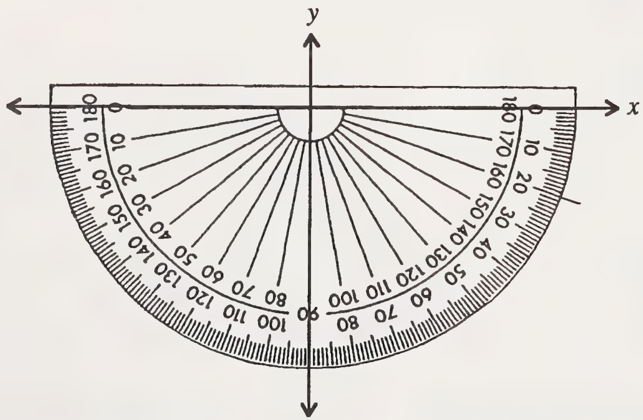


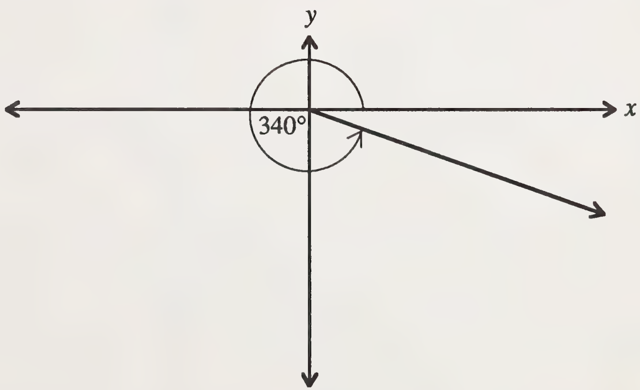
If you do not see how this measurement was made, it will be worthwhile to turn to the **Extra Help** section at the end of this topic.

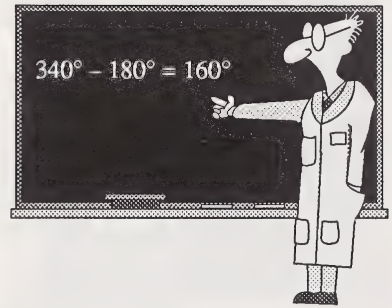
Example 4

Draw a 340° angle in standard position.

Solution:

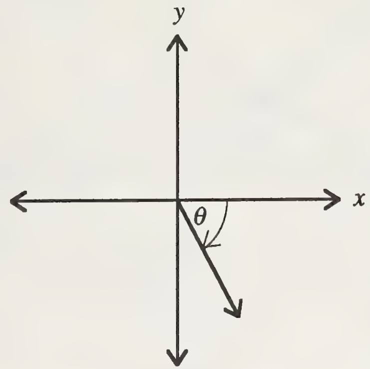


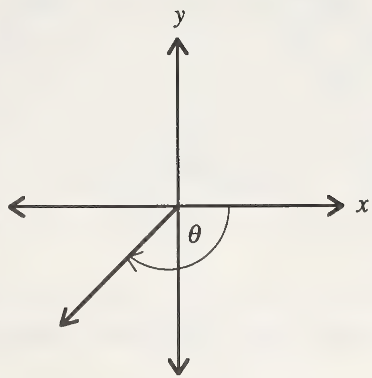


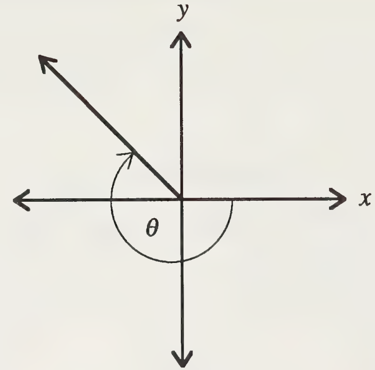


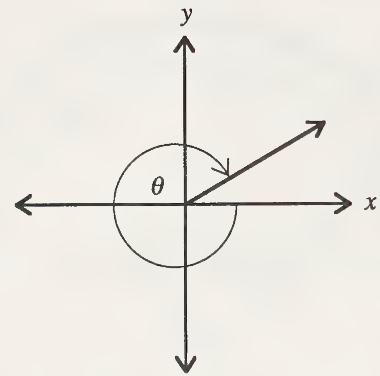
When an angle is measured from the initial arm to the terminal arm in a clockwise direction, the angle is negative.

The following are examples of negative angles in standard position.





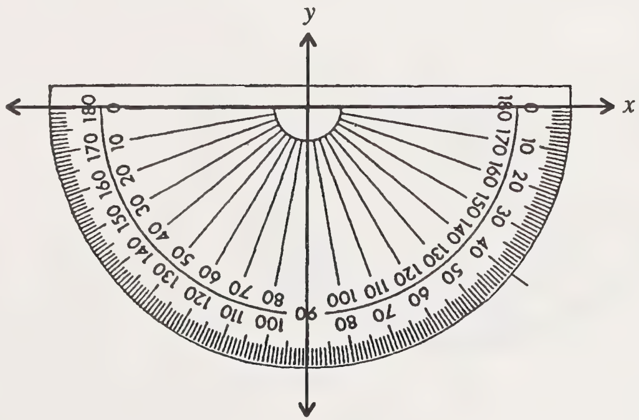


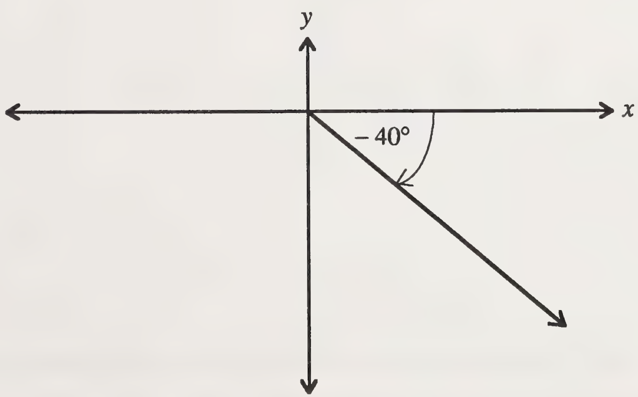


Example 5

Draw a $-40°$ angle in standard position.
Solution:

Measure the angle in a clockwise direction from the initial arm.

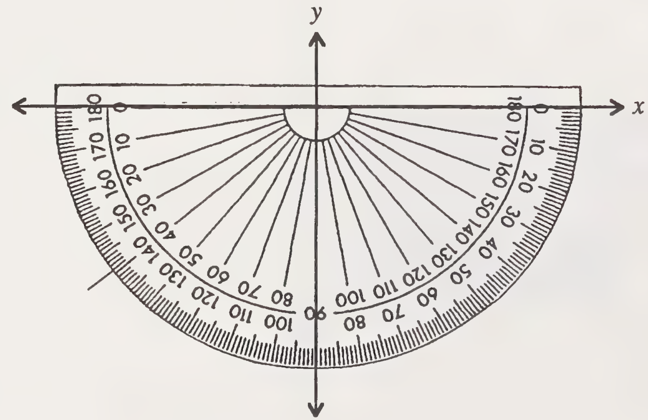


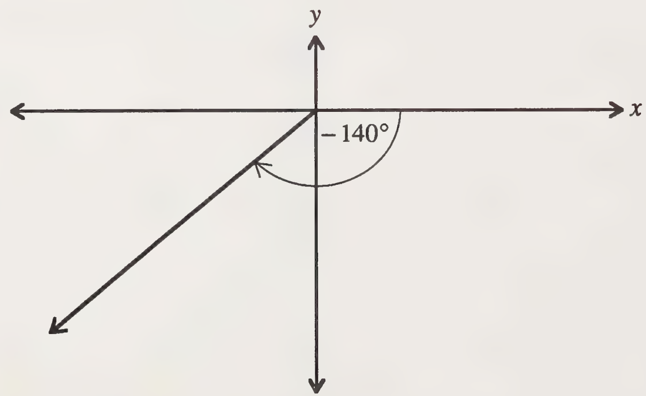


Example 6

Draw a $-140°$ angle in standard position.
Solution:

Measure the angle clockwise from the initial arm.





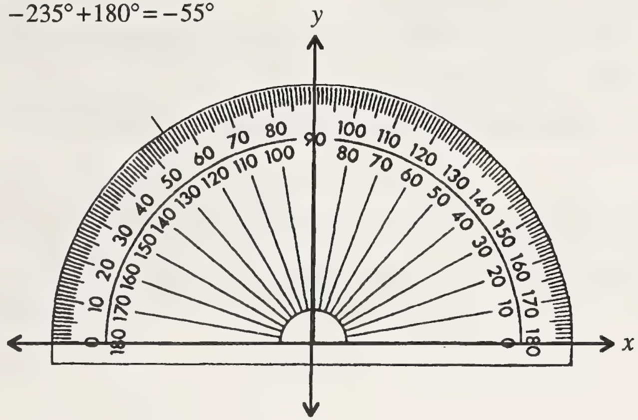
Example 7

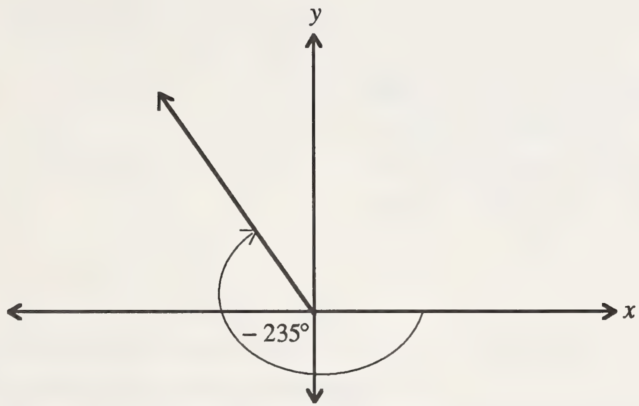
Draw a $-235°$ angle in standard position.

Solution:

This angle is less than $-180°$; therefore, you will have to add $180°$ to the angle to get the portion above the x-axis.

$-235° + 180° = -55°$





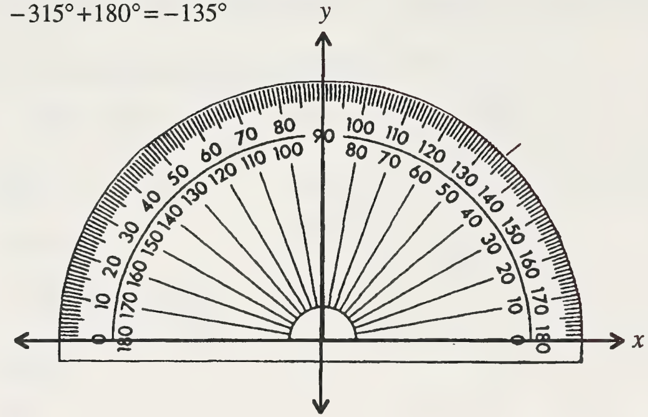
Example 8

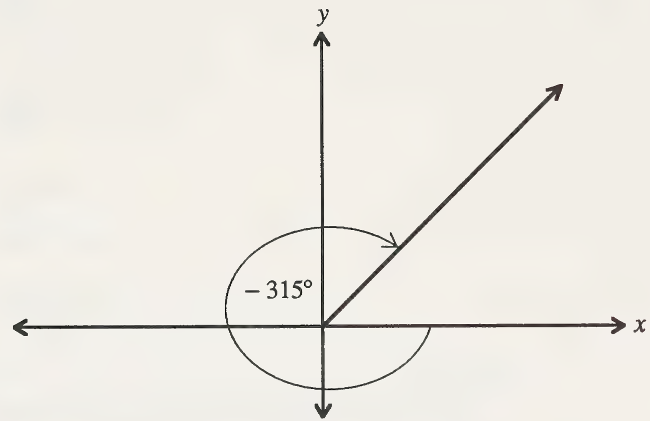
Draw a $-315°$ angle in standard position.

Solution:

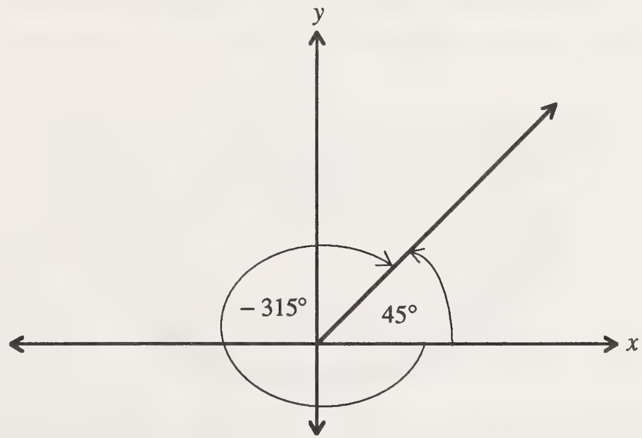
Add $180°$ to the required angle.

$-315° + 180° = -135°$





You may have noticed that there are two angles produced by the terminal arm; one is positive and one is negative. For example, $-315°$ has the same terminal arm as $45°$.



These angles are called **coterminal angles**.

If you are given any angle in standard position, you can find one of its coterminal angles by the following steps.

- If the angle is positive, subtract 360° from the angle to get a coterminal angle.
- If the angle is negative, add 360° to the angle to get a coterminal angle.

Example 9

Find a coterminal angle for each angle that follows.

- 37°

  Solution:

  $$37° - 360° = -323°$$

- – 60°

  Solution:

  $$-60° + 360° = 300°$$

- 121°

  Solution:

  $$121° - 360° = -239°$$

- – 270°

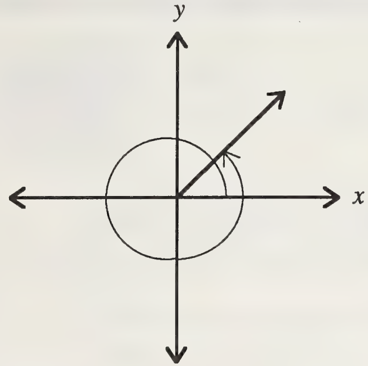
  Solution:

  $$-270° + 360° = 90°$$

Angles can also be greater than 360° or less than $-360°$.

If the terminal arm is rotated more than a full circle in the positive direction (counterclockwise), an angle whose measure is greater than 360 is produced.

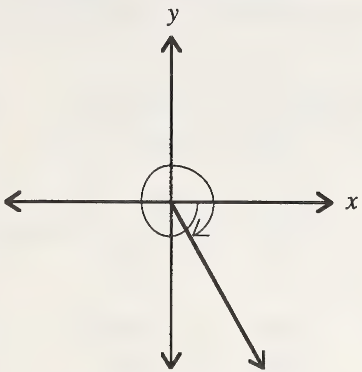
For example, consider the following coordinate plane.



The terminal arm has rotated a complete circle plus 45°. The measure of this angle is $360° + 45° = 405°$.

Since the angles with measures of 45° and 405° have the same initial and terminal arms, they are coterminal. The smallest positive angle for this terminal arm is called the **principal angle**.

An example of an angle less than $-360°$ is shown as follows.



The terminal arm has rotated a complete revolution plus $-60°$. The measure of this angle is $-360° - 60° = -420°$.

Since the angles with measures $-60°$ and $-420°$ have the same initial and terminal arms, they are coterminal.

Example 10

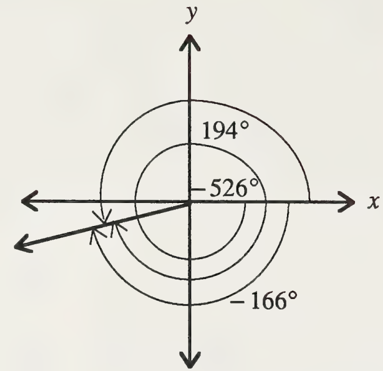
Find two coterminal angles for each of the following.

- $-526°$

  Solution:

  $-526° + 360° = -166°$

  $-166° + 360° = 194°$



- $599°$

  Solution:

  $599° - 360° = 239°$

  $239° - 360° = -121°$

- $895°$

  Solution:

  $895° - 360° = 535°$

  $535° - 360° = 175°$

- $-710°$

  Solution:

  $-710° + 360° = -350°$

  $-350° + 360° = 10°$

There are an infinite number of coterminal angles. Each is separated from its nearest coterminal angle by 360°.

Do one of the following questions.

1. For each of the eight angles given, answer the questions that follow.

125°	400°
−125°	−400°
65°	275°
−65°	−275°

   a. State the quadrant in which the angle will have its terminal arm.
   b. Draw the angle in standard position on a coordinate plane. Graph paper is provided in **Appendix B**.
   c. Find two coterminal angles (one positive, one negative).

2. For each of the eight angles given, answer the questions that follow.

235°	660°
−235°	−395°
25°	110°
−330°	−120°

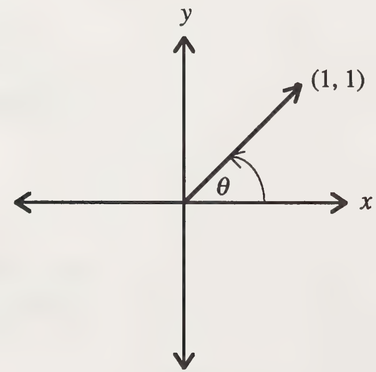
   a. State the quadrant in which the angle will have its terminal arm.
   b. Draw the angle in standard position on a coordinate plane. Graph paper is provided in **Appendix B**.
   c. Find two coterminal angles (one positive, one negative)

For solutions to **Activity 1**, turn to **Appendix A, Topic 2**.

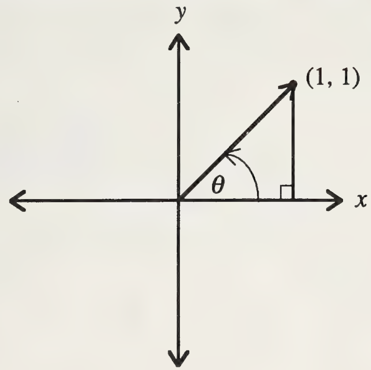
Activity 2

Express the trigonometric ratios and calculate the reference angle for an angle drawn in standard position on a coordinate plane.

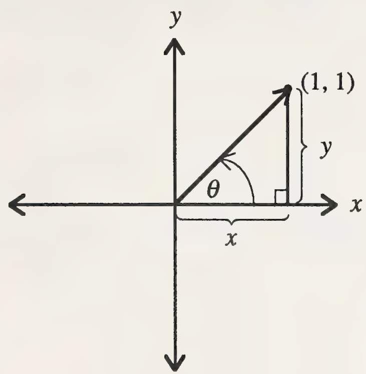
The following are illustrations that relate right-angled triangles with angles drawn in standard position on a coordinate plane.



Draw a perpendicular line from point (1, 1) to the x-axis.



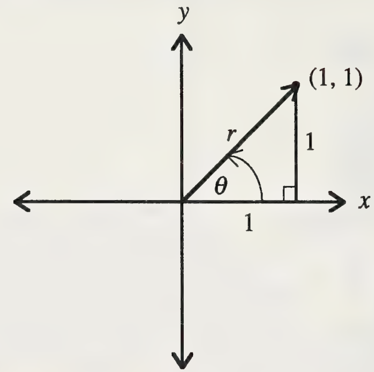
Now you have a right-angled triangle. Because point (1, 1) is known and the triangle lies on the coordinate plane, more information for this triangle can be obtained.



The lengths of the sides of the triangle are x and y. The exact lengths are determined from the point (x, y) which in this case is (1, 1).

$$x = 1$$
$$y = 1$$

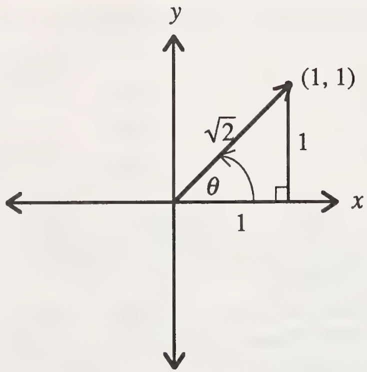
The information gathered so far is shown as follows.



The length of the terminal arm can be calculated using the Pythagorean theorem. The length of the terminal arm is called r, and this length is always positive.

$$r^2 = 1^2 + 1^2$$
$$r^2 = 1 + 1$$
$$r^2 = 2$$
$$r = \sqrt{2}$$

The completely labelled triangle is shown as follows.



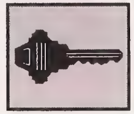
Using the skills learned in Topic 1, you can determine the exact trigonometric ratios of the angles in this triangle.

$$\sin\ \theta = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos\ \theta = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan\ \theta = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{1}{1} = 1$$

A summary for any trigonometric function for any angle is shown. It is worth your while to memorize the trigonometric functions since they will be used throughout this topic.



In general for any angle on a coordinate plane produced when the terminal arm ends at point (x, y), the trigonometric ratios are as follows:

$$\sin\ \theta = \frac{y}{r}$$

$$\cos\ \theta = \frac{x}{r}$$

$$\tan\ \theta = \frac{y}{x}$$

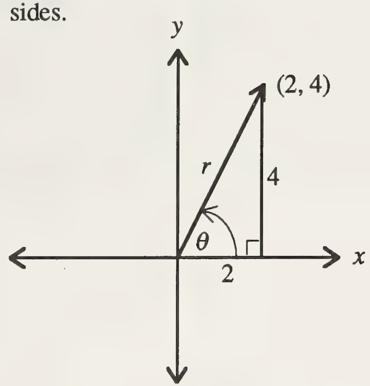
where $r = \sqrt{x^2 + y^2}$

Example 11

Determine the three primary trigonometric ratios for an angle produced by the terminal arm ending at point (2, 4).

Solution:

Step 1: Make a sketch to aid your solution. Draw a perpendicular line from the point to the x-axis and label the sides.



Step 2: Solve for r.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{2^2 + 4^2} \\ &= \sqrt{4+16} \\ &= \sqrt{20} \\ &= 2\sqrt{5} \end{aligned}$$

Step 3: Determine the three primary trigonometric ratios.

$$\begin{aligned} \sin\,\theta &= \frac{y}{r} \\ &= \frac{4}{2\sqrt{5}} \\ &= \frac{2}{\sqrt{5}} \\ &= \frac{2\sqrt{5}}{5} \end{aligned}$$

$$\begin{aligned} \cos\,\theta &= \frac{x}{r} \\ &= \frac{2}{2\sqrt{5}} \\ &= \frac{1}{\sqrt{5}} \\ &= \frac{\sqrt{5}}{5} \end{aligned}$$

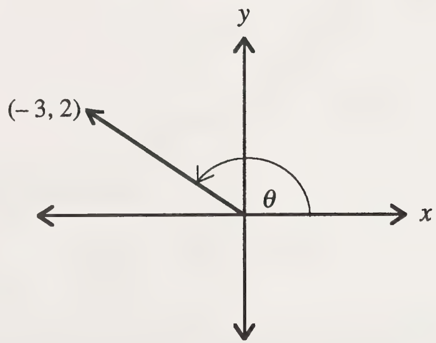
$$\begin{aligned} \tan\,\theta &= \frac{y}{x} \\ &= \frac{4}{2} \\ &= 2 \end{aligned}$$

The perpendicular line is always perpendicular to the x-axis. Do not draw any perpendicular line which is perpendicular to the y-axis.

$$\begin{aligned} \sqrt{20} &= \sqrt{4 \times 5} \\ &= \sqrt{4} \times \sqrt{5} \\ &= 2\sqrt{5} \end{aligned}$$

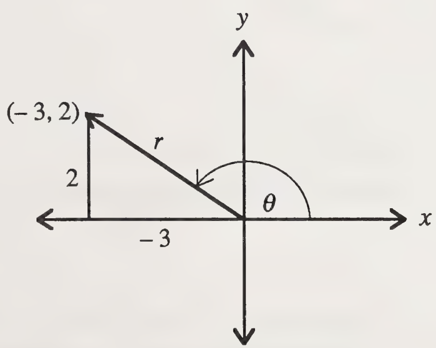
Example 12

Determine the primary trigonometric ratios for θ in the diagram shown.



Solution:

Step 1: Sketch the triangle.



Step 2: Solve for *r*.

$$
\begin{aligned}
r &= \sqrt{(-3)^2 + (2)^2} \\
&= \sqrt{9+4} \\
&= \sqrt{13}
\end{aligned}
$$

Step 3: Determine the trigonometric ratios.

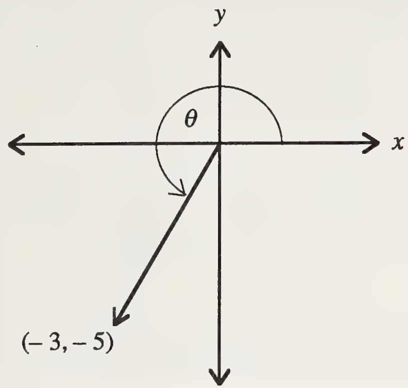
$$
\begin{aligned}
\sin\ \theta &= \frac{2}{\sqrt{13}} \\
&= \frac{2\sqrt{13}}{13}
\end{aligned}
$$

$$
\begin{aligned}
\cos\ \theta &= \frac{-3}{\sqrt{13}} \\
&= \frac{-3\sqrt{13}}{13}
\end{aligned}
$$

$$
\begin{aligned}
\tan\ \theta &= \frac{2}{-3} \\
&= -\frac{2}{3}
\end{aligned}
$$

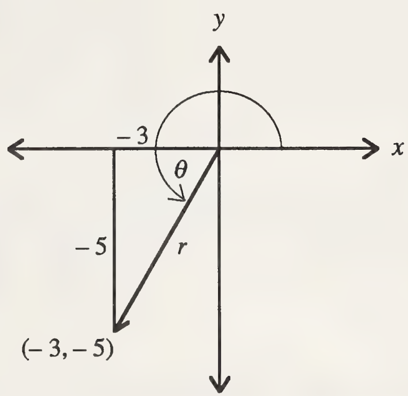
Example 13

Determine the trigonometric ratios for the angle shown.



Solution:

Step 1:



Step 2: $r=\sqrt{(-3)^2+(-5)^2}$

$r=\sqrt{9+25}$

$r=\sqrt{34}$

Step 3: $\sin\ \theta=\frac{-5}{\sqrt{34}}$

$=\frac{-5\sqrt{34}}{34}$

$\cos\ \theta=\frac{-3}{\sqrt{34}}$

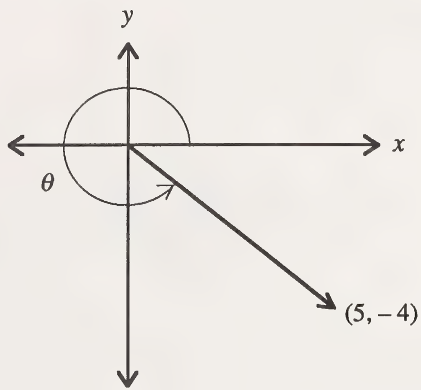
$=\frac{-3\sqrt{34}}{34}$

$\tan\ \theta=\frac{-5}{-3}$

$=\frac{5}{3}$

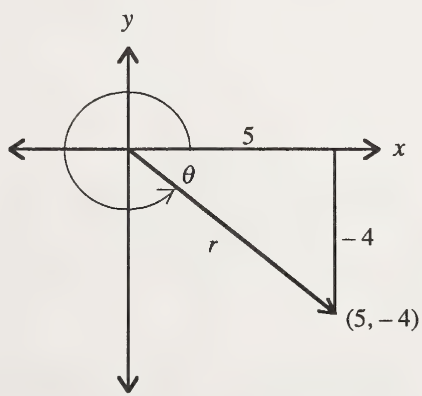
Example 14

Determine the trigonometric ratios for the following angle.



Solution:

Step 1:



Step 2: $r = \sqrt{(5)^2 + (-4)^2}$

$r = \sqrt{25+16}$

$r = \sqrt{41}$

Step 3: $\sin\ \theta = \dfrac{-4}{\sqrt{41}}$

$= \dfrac{-4\sqrt{41}}{41}$

$\cos\ \theta = \dfrac{5}{\sqrt{41}}$

$= \dfrac{5\sqrt{41}}{41}$

$\tan\ \theta = \dfrac{-4}{5}$

Do one of the following questions.

1. Determine the three primary trigonometric ratios for the angles which are in standard position when the terminal arm ends at the given point.

   a. $(-1, 5)$

   b. $(2, -4)$

   c. $(-3, -3)$

   d. $(1, 6)$

2. Determine the three primary trigonometric ratios for the angles which are in standard position when the terminal arm ends at the given point.

   a. $(3, 5)$

   b. $(-4, 2)$

   c. $(-4, -2)$

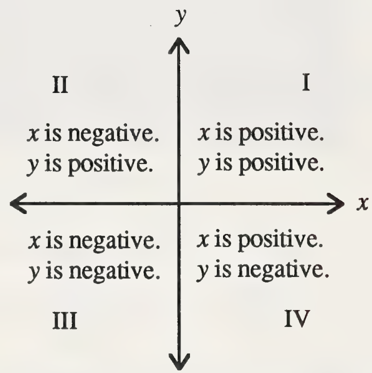
   d. $(2, -3)$

For solutions to **Activity 2**, turn to **Appendix A, Topic 2**.

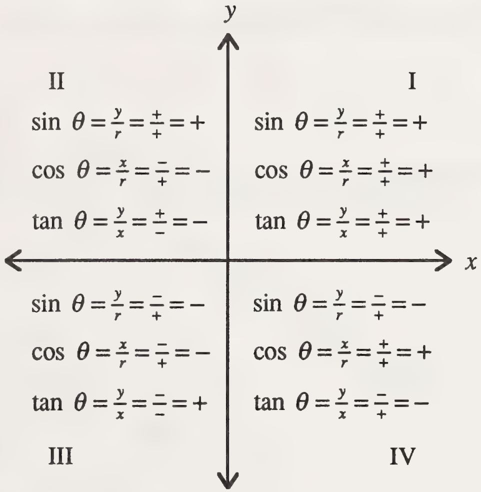
Some generalizations about the sign of the value of the primary trigonometric functions in each quadrant can be drawn.

The length of the terminal arm (r) is always positive since $r = \sqrt{x^2 + y^2}$.

The signs of x and y are summarized in the following table.



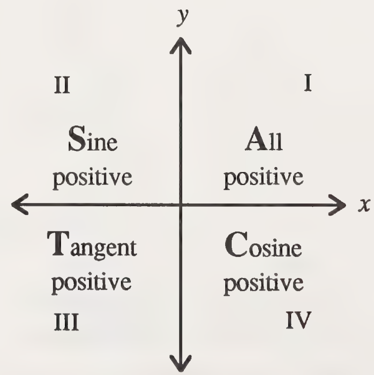
Since the three primary trigonometric functions are defined in terms of x, y, and r, they are written as follows:



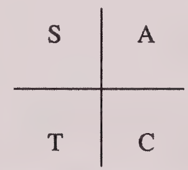
The following chart illustrates the sign of the primary trigonometric functions in each quadrant.

Quadrant II	Quadrant I
sin +	sin +
cos –	cos +
tan –	tan +
Quadrant III	**Quadrant IV**
sin –	sin –
cos –	cos +
tan +	tan –

The following diagram shows the primary trigonometric functions that are **positive** in each quadrant. In Quadrant I all trigonometric functions are positive. In Quadrant II the sine ratio is positive. In Quadrant III the tangent ratio is positive. In Quadrant IV the cosine ratio is positive.



This memory aid is called the **CAST rule.**



Try one of the following questions.

3. a. State the sign of each of the following trigonometric functions.

   i. $\cos\theta$ in Quadrant III

   ii. $\tan\theta$ in Quadrant IV

   iii. $\sin\theta$ in Quadrant II

   b. State the quadrant in which the terminal arm lies so that the conditions are satisfied.

   i. $\cos\theta$ and $\tan\theta$ are positive.

   ii. $\cos\theta$ and $\sin\theta$ are negative.

   iii. $\sin\theta$ is positive and $\tan\theta$ is negative.

   iv. $\cos\theta$ is negative and $\tan\theta$ is positive.

   c. State the sign of each function.

   i. $\cos 212°$ ii. $\tan 181°$

   iii. $\sin 236°$ iv. $\cos 675°$

   v. $\tan 317°$ vi. $\sin -62°$

   vii. $\cos 174°$ viii. $\tan -295°$

   ix. $\sin 39°$ x. $\sin -543°$

4. a. State the sign of each of the following trigonometric functions.

   i. $\sin\theta$ in Quadrant III

   ii. $\cos\theta$ in Quadrant IV

   iii. $\tan\theta$ in Quadrant II

   b. State the quadrant in which the terminal arm lies so that the conditions are satisfied.

   i. $\cos\theta$ and $\tan\theta$ are negative.

   ii. $\sin\theta$ and $\tan\theta$ are negative.

   iii. $\sin\theta$ is negative and $\tan\theta$ is positive.

   iv. $\cos\theta$ is positive and $\sin\theta$ is negative.

c. State the sign of each function.

i. $\cos 43°$

ii. $\tan 312°$

iii. $\sin 499°$

iv. $\cos 155°$

v. $\tan -260°$

vi. $\sin -11°$

vii. $\cos -127°$

viii. $\tan 163°$

ix. $\sin 260°$

x. $\sin 359°$

For solutions to **Activity 2**, turn to **Appendix A, Topic 2**.

The basic trigonometric functions (that is, sine, cosine, and tangent) were found to have exact values for 30°, 45°, and 60°. You verified the values using your calculator. Here is a little puzzle that you can try to solve using your calculator.

Enter	Display
45	45
$\boxed{\text{sin}}$	0.707106781

Enter	Display
135	135
$\boxed{\text{sin}}$	0.707106781

Enter	Display
225	225
$\boxed{\text{sin}}$	−0.707106781

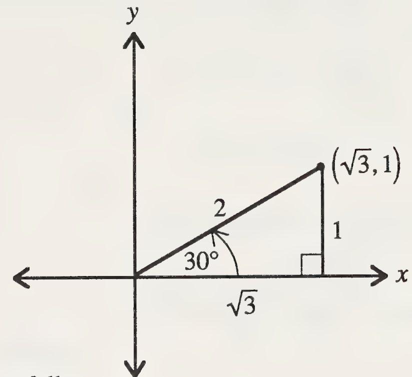
Enter	Display
315	315
$\boxed{\text{sin}}$	−0.707106781

You can see that the absolute values of $\sin 45°$, $\sin 135°$, $\sin 225°$, and $\sin 315°$ are all equal. The 45° acute angle is called the **reference angle** of 135°, 225°, and 315°. A reference angle is also called a related angle. The other trigonometric functions follow the same pattern. In other words, this applies to the tangent and cosine functions as well as the sine function.

If θ is the acute reference angle in the first quadrant, the other angles in the other quadrants can be determined by subtracting θ from 180° $(180° - \theta)$, adding θ to 180° $(180° + \theta)$, subtracting θ from 360° $(360° - \theta)$, or adding $(n \times 360°)$ to any one of the four angles mentioned.

Now you will examine the relationships between angles in Quadrants II, III, and IV compared with their reference angle in Quadrant I.

Construct a 30° - 60° - 90° triangle in standard position in the first quadrant.



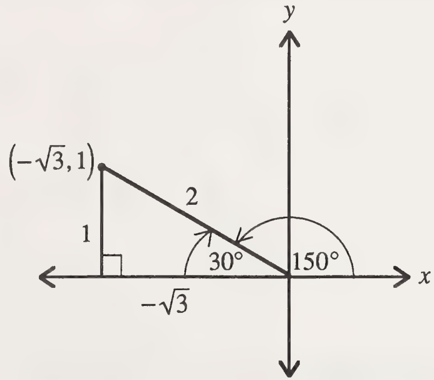
The three trigonometric ratios for 30° are as follows:

$$\sin 30° = \frac{1}{2}$$

$$\cos 30° = \frac{\sqrt{3}}{2}$$

$$\tan 30° = \frac{1}{\sqrt{3}}$$
$$= \frac{\sqrt{3}}{3}$$

Construct a 30° - 60° - 90° triangle in the second quadrant.



The two angles which result are 30° and 150°. The 150° angle is calculated since this is the angle in standard position.

Notice that $180° - 30° = 150°$.

The trigonometric ratios for 150° are as follows.

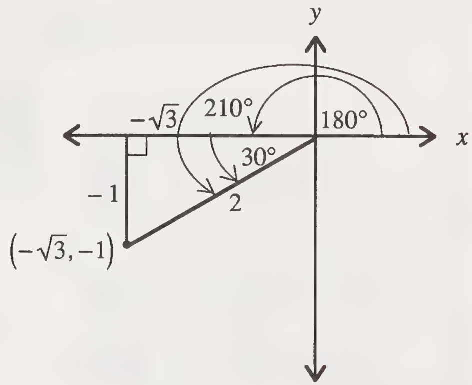
$$\sin 150° = \frac{1}{2}$$

$$\cos 150° = \frac{-\sqrt{3}}{2}$$

$$\tan 150° = \frac{-1}{\sqrt{3}}$$

$$= \frac{-\sqrt{3}}{3}$$

Next construct a 30° - 60° - 90° triangle in the third quadrant.



The angle in standard position is a 210° angle since this angle is the sum of 180° and 30°.

The trigonometric ratios for a 210° angle are as follows:

$$\sin 210° = \frac{-1}{2}$$

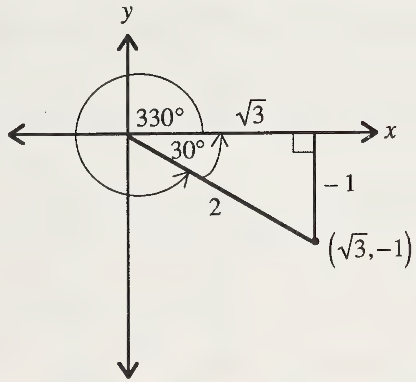
$$\cos 210° = \frac{-\sqrt{3}}{2}$$

$$\tan 210° = \frac{-1}{-\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}$$

$$= \frac{\sqrt{3}}{3}$$

Construct a 30° - 60° - 90° triangle in the fourth quadrant.



The trigonometric ratios for a 330° angle are as follows:

$$\sin 330° = \frac{-1}{2}$$

$$\cos 330° = \frac{\sqrt{3}}{2}$$

$$\tan 330° = \frac{-1}{\sqrt{3}}$$
$$= \frac{-\sqrt{3}}{3}$$

The following is a summary of the information derived from each quadrant.

Quadrant I

$$\sin 30° = \frac{1}{2}$$
$$\cos 30° = \frac{\sqrt{3}}{2}$$
$$\tan 30° = \frac{1}{\sqrt{3}}$$
$$= \frac{\sqrt{3}}{3}$$

Quadrant II

$$\sin 150° = \frac{1}{2}$$
$$\cos 150° = \frac{-\sqrt{3}}{2}$$
$$\tan 150° = \frac{-1}{\sqrt{3}}$$
$$= \frac{-\sqrt{3}}{3}$$

Quadrant III

$$\sin 210° = -\frac{1}{2}$$
$$\cos 210° = \frac{-\sqrt{3}}{2}$$
$$\tan 210° = \frac{1}{\sqrt{3}}$$
$$= \frac{\sqrt{3}}{3}$$

Quadrant IV

$$\sin 330° = -\frac{1}{2}$$
$$\cos 330° = \frac{\sqrt{3}}{2}$$
$$\tan 330° = \frac{-1}{\sqrt{3}}$$
$$= \frac{-\sqrt{3}}{3}$$

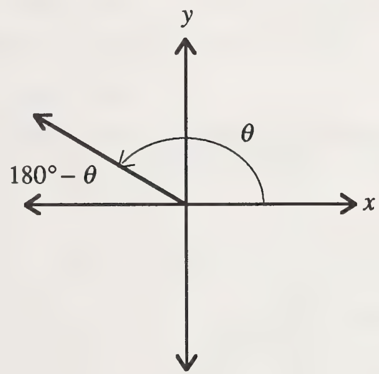
Notice the absolute value of each function is the same for all four quadrants. Only the sign varies in each quadrant.

The first quadrant angle (30° in the example) can be used to determine the absolute value of the other angles (150°, 210°, and 330°) in the other quadrants.

All that needs to be added is the correct sign.

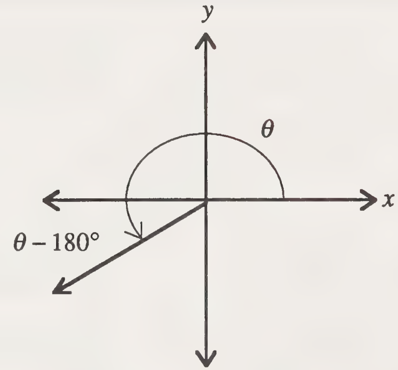
In general the first quadrant angle is called the **reference angle** which is used to determine the trigonometric ratios for related angles in the other quadrants. The way you calculate the reference angle is as follows.

Quadrant II



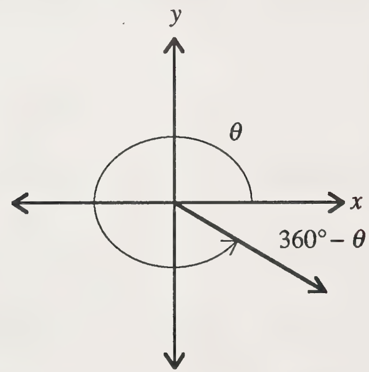
The reference angle is equal to $180° - \theta$ since $90° < \theta \leq 180°$.

Quadrant III



The reference angle is equal to $\theta - 180°$ since $180° < \theta \leq 270°$.

Quadrant IV



The reference angle is equal to $360° - \theta$ since $270° < \theta \leq 360°$.

Example 15

Determine the reference angle for each of the following angles.

- $174°$

Solution:

$174°$ is between $90°$ and $180°$; therefore, you use the formula $180° - \theta$ to obtain the reference angle.

$$\begin{aligned}\text{Reference angle} &= 180° - 174° \\ &= 6°\end{aligned}$$

- $217°$

Solution:

$217°$ is between $180°$ and $270°$; therefore, you use $\theta - 180°$.

$$\begin{aligned}\text{Reference angle} &= 217° - 180° \\ &= 37°\end{aligned}$$

- $312°$

Solution:

$312°$ is between $270°$ and $360°$; therefore, you use $360° - \theta$.

$$\begin{aligned}\text{Reference angle} &= 360° - 312° \\ &= 48°\end{aligned}$$

- $-50°$

Solution:

$-50°$ is not in the range of $0°$ to $360°$; therefore, you must first find a coterminal angle which is in the correct range.

The coterminal angle for $-50°$ is $310°$ $(-50° + 360° = 310°)$.

$$\begin{aligned}\text{Reference angle} &= 360° - 310° \\ &= 50°\end{aligned}$$

- $500°$

Solution:

The coterminal angle for $500°$ is $140° (500° - 360° = 140°)$.

$$\begin{aligned}\text{Reference angle} &= 180° - 140° \\ &= 40°\end{aligned}$$

- $-260°$

Solution:

The coterminal angle for $-260°$ is $100°$ $[360° + (-260°) = 100°]$.

$$\begin{aligned}\text{Reference angle} &= 180° - 100° \\ &= 80°\end{aligned}$$

You can now combine the reference-angle calculations with the sign of the trigonometric ratios for each quadrant. The following is a summary table.

II S	I A
$\sin\theta = \sin(180° - \theta)$	$\sin\theta$
$\cos\theta = -\cos(180° - \theta)$	$\cos\theta$
$\tan\theta = -\tan(180° - \theta)$	$\tan\theta$
III T	**IV C**
$\sin\theta = -\sin(\theta - 180°)$	$\sin\theta = -\sin(360° - \theta)$
$\cos\theta = -\cos(\theta - 180°)$	$\cos\theta = \cos(360° - \theta)$
$\tan\theta = \tan(\theta - 180°)$	$\tan\theta = -\tan(360° - \theta)$

Do the following exercises.

5. What is the reference angle for the following? Use a calculator to show that the absolute value of the sine of each angle is equal to the absolute value of the sine of its reference angle.

   a. 125° b. 65°

   c. 350° d. 230°

6. Determine the value of the three primary trigonometric functions for each angle in standard position. Leave your answer as a function of the reference angle.
   For example, $\cos 150° = -\cos 30°$.

   a. 98° b. 263°

   c. 345° d. −77°

   e. −488° f. 569°

   g. 49°

7. Determine the value of the three primary trigonometric functions for each angle in standard position. Leave your answer as a function of the reference angle.
   For example, $\sin 295° = -\sin 65°$.

   a. 63° b. 149°

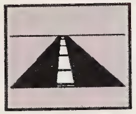
   c. 186° d. 349°

   e. −20° f. −592°

   g. −320°

For solutions to **Activity 2**, turn to **Appendix A, Topic 2**.

Activity 3



Calculate the trigonometric ratios for any angle.

You will calculate the trigonometric ratios for any angle using tables and calculators. Most scientific calculators have the three primary trigonometric functions available. The table of trigonometric functions in **Appendix B** gives you the values of the six trigonometric functions when the measure of the angle is a whole number. If the measure of an angle is not a whole number, then a different table will be needed.

The following examples show you how to use the trigonometric table to determine the trigonometric ratios.

Example 16

Determine the six trigonometric ratios for the following angles. Use the given trigonometric table in **Appendix B**. The trigonometric ratios are read directly from the table.

- $\theta = 9°$

  Solution:

  $\sin 9° \doteq 0.1564$

  $\cos 9° \doteq 0.9877$

  $\tan 9° \doteq 0.1584$

  $\csc 9° \doteq 6.3925$

  $\sec 9° \doteq 1.0125$

  $\cot 9° \doteq 6.3138$

- $\theta = 90°$

  Solution:

  $\sin 90° \doteq 1$

  $\cos 90° \doteq 0$

  $\tan 90° \doteq$ undefined

  $\csc 90° \doteq 1$

  $\sec 90° \doteq$ undefined

  $\cot 90° \doteq 0$

- $\theta = 67°$

  Solution:

  $\sin 67° \doteq 0.9205$

  $\cos 67° \doteq 0.3907$

  $\tan 67° \doteq 2.3559$

  $\csc 67° \doteq 1.0864$

  $\sec 67° \doteq 2.5593$

  $\cot 67° \doteq 0.4245$

- $\theta = 0°$

  Solution:

  $\sin 0° = 0$

  $\cos 0° = 1$

  $\tan 0° = 0$

  $\csc 0° =$ undefined

  $\sec 0° = 1$

  $\cot 0° =$ undefined

Since you are dealing with the trigonometric ratios of any angle, the skills and concepts of the previous activities are essential.

The following are steps for finding the trigonometric ratios for any angle using the trigonometric table.

Step 1: Calculate the reference angle for the given angle.

Step 2: Assign the correct sign to the trigonometric function so that it corresponds to the quadrant in which the terminal arm is located.

Step 3: Read the value from the trigonometric table.

Example 17

Using your trigonometric table in **Appendix B**, determine all three primary trigonometric ratios for the following angles.

- 111°

Solution:

$$\begin{aligned}\sin 111° &= \sin(180° - 111°)\\ &= \sin 69°\\ &\doteq 0.9336\end{aligned}$$

$$\begin{aligned}\cos 111° &= -\cos(180° - 111°)\\ &= -\cos 69°\\ &\doteq -0.3584\end{aligned}$$

$$\begin{aligned}\tan 111° &= -\tan(180° - 111°)\\ &= -\tan 69°\\ &\doteq -2.6051\end{aligned}$$

- 216°

Solution:

$$\begin{aligned}\sin 216° &= -\sin(216° - 180°)\\ &= -\sin 36°\\ &\doteq -0.5878\\ \cos 216° &= -\cos(216° - 180°)\\ &= -\cos 36°\\ &\doteq -0.8090\\ \tan 216° &= \tan(216° - 180°)\\ &= \tan 36°\\ &\doteq 0.7265\end{aligned}$$

- 275°

Solution:

$$\begin{aligned}\sin 275° &= -\sin(360° - 275°)\\ &= -\sin 85°\\ &\doteq -0.9962\\ \cos 275° &= \cos(360° - 275°)\\ &= \cos 85°\\ &\doteq 0.0872\\ \tan 275° &= -\tan(360° - 275°)\\ &= -\tan 85°\\ &\doteq -11.4301\end{aligned}$$

- 508°

Solution:

508° is coterminal with 148°.

$$\begin{aligned} \sin 508° &= \sin(180° - 148°) \\ &= \sin 32° \\ &\doteq 0.5299 \end{aligned}$$

$$\begin{aligned} \cos 508° &= -\cos(180° - 148°) \\ &= -\cos 32° \\ &\doteq -0.8480 \end{aligned}$$

$$\begin{aligned} \tan 508° &= -\tan(180° - 148°) \\ &= -\tan 32° \\ &\doteq -0.6249 \end{aligned}$$

- −29°

Solution:

−29° is coterminal with 331°.

$$\begin{aligned} \sin(-29°) &= -\sin(360° - 331°) \\ &= -\sin 29° \\ &\doteq -0.4848 \end{aligned}$$

$$\begin{aligned} \cos(-29°) &= \cos(360° - 331°) \\ &= \cos 29° \\ &\doteq 0.8746 \end{aligned}$$

$$\begin{aligned} \tan(-29°) &= -\tan(360° - 331°) \\ &= -\tan 29° \\ &\doteq -0.5543 \end{aligned}$$

- $-127°$

Solution:

$$\begin{aligned}\sin -127° &= \sin 233° \\ &= -\sin(233° - 180°) \\ &= -\sin 53° \\ &\doteq -0.7986\end{aligned}$$

$$\begin{aligned}\cos -127° &= \cos 233° \\ &= -\cos(233° - 180°) \\ &= -\cos 53° \\ &\doteq -0.6018\end{aligned}$$

$$\begin{aligned}\tan -127° &= \tan 233° \\ &= \tan(233° - 180°) \\ &= \tan 53° \\ &\doteq 1.3270\end{aligned}$$

Try one of the following questions.

1. Using the trigonometric table in **Appendix B**, determine the three primary trigonometric ratios for the following angles.

   a. 163° b. 92°

   c. 270° d. −321°

   e. 660° f. 23°

2. Use the trigonometric table in **Appendix B** to determine the three primary trigonometric ratios for the following angles.

   a. 31° b. −31°

   c. 180° d. 193°

   e. 491° f. −375°

For solutions to **Activity 3**, turn to **Appendix A, Topic 2**.

A calculator that has trigonometric functions makes these calculations very simple.

Example 18

Use your calculator to determine the trigonometric ratios for the following angles.

- 45°

Solution:

Enter	Display
45	45
sin	0.707106781

Enter	Display
45	45
cos	0.707106781

Enter	Display
45	45
tan	1

These values correspond to the values given in the table. Check to see that they correspond.

- 130°

Solution:

Enter	Display
130	130
sin	0.766044443

Enter	Display
130	130
cos	– 0.642787609

Enter	Display
130	130
tan	– 1.191753593

Did you notice that it is not necessary to find the reference angle?

• $-74°$

Solution:

Enter	Display
74	74
+/–	–74
sin	–0.961261696

Enter	Display
74	74
+/–	–74
cos	0.275637355

Enter	Display
74	74
+/–	–74
tan	–3.487414444

Try finding the tangent of 90°. You should get an error display. This means you have performed an impossible operation. From the table you see the tangent of 90° is undefined. What has occurred here is an attempt to divide by zero which is not allowed. If you get an error message, remember that you have an undefined answer.

Try one of the following questions.

3. Use your calculator to determine the three primary trigonometric rtios for each angle.

   a. –647° b. 317°

   c. 21° d. 180°

   e. 217°

4. Use your calculator to determine the three primary trigonometric ratios for each angle.

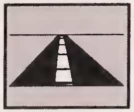
   a. 379° b. –95°

   c. 275° d. 183°

   e. 175°

For solutions to **Activity 3**, turn to **Appendix A, Topic 2**.

Activity 4



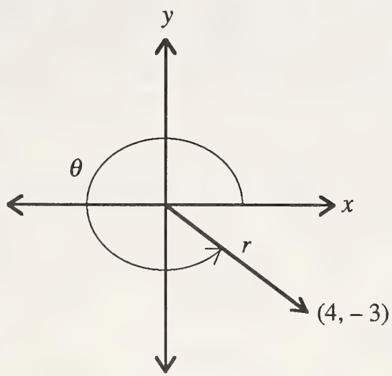
Determine any two values of x, y, r, and θ given the other two.

The relationship between x, y, r, and θ was examined in Activity 2. You learned to calculate the trigonometric ratios for any angle in Activity 3. In this activity you will combine these skills to solve for the unknown variables x, y, r, and θ given any two of them.

There are six possible cases. You will examine each case in detail. You should not try to memorize these cases, but rather try to understand the underlying principles that will enable you to solve them.

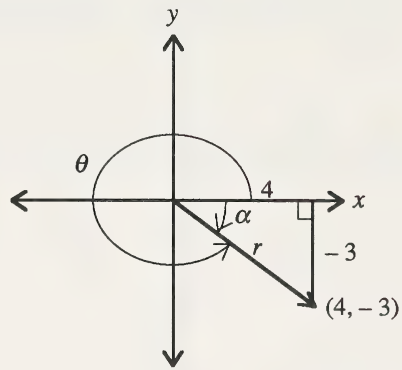
Case 1

You are given x and y and you are required to find r and θ.



Start by drawing a line from the terminal point so that it is perpendicular to the x-axis. This will make a right-angled triangle such that the sides of the triangle are the x-axis, the perpendicular line, and the terminal arm. The coordinates of the ordered pairs will give the lengths of the two sides of the triangle. The side along the x-axis is 4 units and the perpendicular side is – 3 units. (The negative sign simply means that you are measuring down from the x-axis).

Since this is a right-angled triangle, you can use the Pythagorean theorem to solve for r.



$$r^2 = x^2 + y^2$$
$$r^2 = 4^2 + (-3)^2$$
$$r^2 = 16 + 9$$
$$r^2 = 25$$
$$r = 5$$

Any one of the three trigonometric functions can be used to find θ.

$$\sin \alpha = \frac{3}{5} \qquad \cos \alpha = \frac{4}{5} \qquad \tan \alpha = \frac{3}{4}$$

$$\sin \alpha = 0.6 \qquad \cos \alpha = 0.8 \qquad \tan \alpha = 0.75$$

$$\alpha \doteq 37° \qquad \alpha \doteq 37° \qquad \alpha \doteq 37°$$

All three ratios determine the same degree value.

What you have found is the reference angle. The reference angle is always measured away from the x-axis and must be between 0° and 90°. Therefore, the x- and y-values must be positive. The measure of the angle in standard position can be found using the rules from Activity 2.

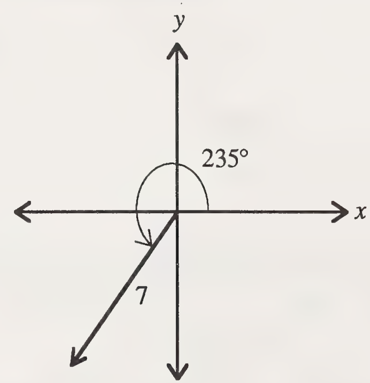
Since α is an angle in Quadrant IV, use the following relationship.

$$\begin{aligned} \theta &= 360° - \alpha \\ &\doteq 360° - 37° \\ &\doteq 323° \end{aligned}$$

The sign of the three ratios for angle θ can also be used to establish the location of the angle. Quadrant IV is the only place where the sine and the tangent are negative and the cosine is positive.

Case 2

You are given θ and r and are required to find x and y.



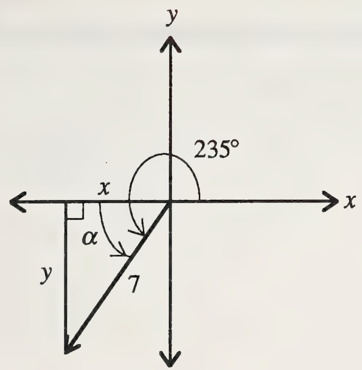
In this case you first find the reference angle α. Since this angle is in Quadrant III, use the following relationship.

$$\theta = 180° + \alpha$$

Fill in the angle measure for θ and solve for α.

$$\begin{aligned} 235° &= 180° + \alpha \\ \alpha &= 235° - 180° \\ \alpha &= 55° \end{aligned}$$

The reference angle for this case is 55°.



Now draw a perpendicular line from the endpoint of the terminal arm to the x-axis.

You now have a right-angled triangle with sides x and y.

The sine and cosine ratios are used to solve for these two sides.

$$-\sin 55° = \frac{y}{7}$$
$$y = 7 \times (-\sin 55°)$$
$$y \doteq 7 \times (-0.8192)$$
$$y \doteq -5.7$$

$$-\cos 55° = \frac{x}{7}$$
$$x = 7 \times (-\cos 55°)$$
$$x \doteq 7 \times (-0.5736)$$
$$x \doteq -4.0$$

Solved to one decimal point, the values for x and y are – 4.0 and – 5.7, respectively.

This case can also be solved using a calculator. This will allow you to find x and y without finding the reference angle.

$$\sin 235° = \frac{y}{7}$$
$$y = 7 \times (\sin 235°)$$
$$y \doteq 7 \times (-0.819\,152\,04)$$
$$y \doteq -5.7$$

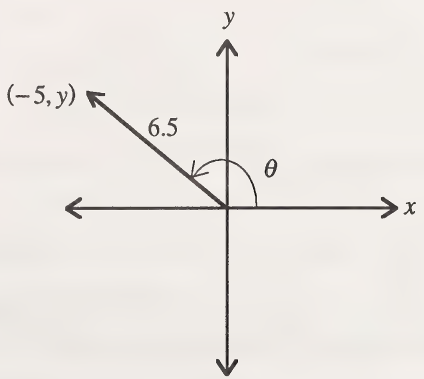
$$\cos 235° = \frac{x}{7}$$
$$x = 7 \times (\cos 235°)$$
$$x \doteq 7 \times (-0.573\,576\,436)$$
$$x \doteq -4.0$$

Either method will yield the same solution.

Both sine and cosine are negative in Quadrant III.

Case 3

You are given x and r and are required to find y and θ.

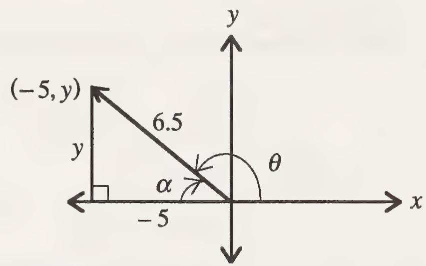


Draw a perpendicular line from the endpoint of the terminal arm to the x-axis. This will form a right-angled triangle.

The hypotenuse has a value of 6.5 and the side along the x-axis has a value of -5. Use the Pythagorean theorem to find y.

$$
\begin{aligned}
r^2 &= x^2 + y^2 \\
y^2 &= r^2 - x^2 \\
y^2 &= (6.5)^2 - (-5)^2 \\
y^2 &= 42.25 - 25 \\
y^2 &= 17.25 \\
y &\doteq 4.2 \quad (y \text{ is positive in Quadrant II.})
\end{aligned}
$$

The length of the other side of the triangle is 4.2 units, rounded to one decimal place.



The angle in standard position (θ) can only be found by first finding the reference angle α. When finding the reference angle, always use the trigonometric function that will use the two given values. For this case, use the cosine function. For the reference angle α, x and y are always positive.

$$
\begin{aligned}
\cos\ \alpha &= \frac{5}{6.5} \\
\cos\ \alpha &\doteq 0.7692 \\
\alpha &\doteq 40^\circ
\end{aligned}
$$

Since this is a second quadrant angle, use the following formula to find θ.

$$
\begin{aligned}
\theta &= 180^\circ - \alpha \\
\theta &= 180^\circ - 40^\circ \\
\theta &= 140^\circ
\end{aligned}
$$

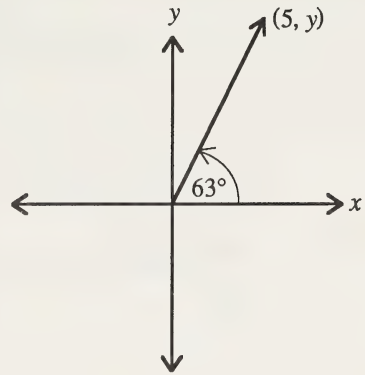
The measure of the angle in standard position is 140°.

If you use a calculator, you can enter 0.7692 → $\boxed{+/-}$ → $\boxed{\text{Inv}}$ → $\boxed{\text{cos}}$ and the final display will be 140°.

Case 4

You are given x and θ and are required to find y and r.

Once one of the sides is found, the other side can be found using the other trigonometric function or the Pythagorean theorem.



Draw a perpendicular line from the endpoint of the terminal arm to the x-axis. This will form the right-angled triangle.

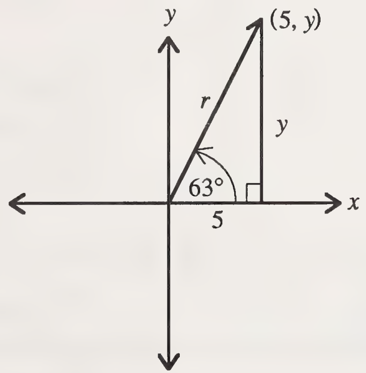
First find y using the tangent function.

$$\tan 63° = \frac{y}{5}$$
$$y = 5 \times \tan 63°$$
$$y \doteq 5 \times 1.9626$$
$$y \doteq 9.8$$

The y-value is 9.8 units.

In this case you are not required to find the reference angle since you are working in the first quadrant.

Now r can be found using either the cosine function or the Pythagorean theorem. Both will be shown here.



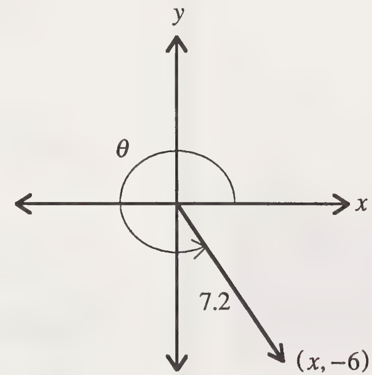
$$\cos 63° = \frac{5}{r}$$
$$r = \frac{5}{\cos 63°}$$
$$r \doteq \frac{5}{0.4540}$$
$$r \doteq 11.0$$

$$r^2 = x^2 + y^2$$
$$r^2 \doteq 5^2 + 9.8^2$$
$$r^2 \doteq 25 + 96.04$$
$$r^2 \doteq 121.04$$
$$r \doteq 11.0$$

The hypotenuse or r is 11.0 units.

Case 5

You are given y and r and you are required to find x and θ.



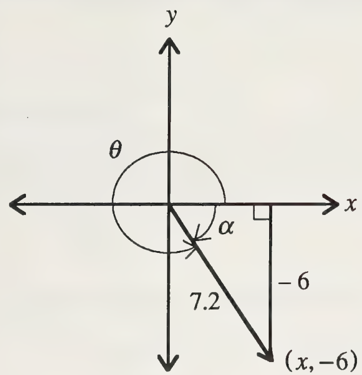
Draw a perpendicular line from the endpoint of the terminal arm to the x-axis. This will form the right-angled triangle.

First use the Pythagorean theorem to find the third side of the triangle.

$$r^2 = x^2 + y^2$$
$$x^2 = r^2 - y^2$$
$$x^2 = 7.2^2 - (-6)^2$$
$$x^2 = 51.84 - 36$$
$$x^2 = 15.84$$
$$x \doteq 4.0$$

The x-value is 4.0 rounded to one decimal place.

The reference angle can be found using the sine function.



Remember that x and y must be positive from angle α.

$$\sin\ \alpha = \frac{6}{7.2}$$
$$\sin\ \alpha \doteq 0.8333$$
$$\alpha \doteq 56°$$

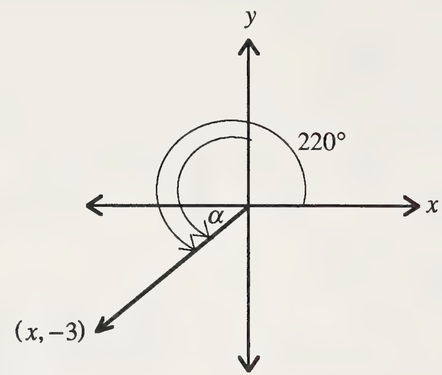
Since this angle is in Quadrant IV, you can solve for θ.

$$\theta = 360° - \alpha$$
$$\theta = 360° - 56°$$
$$\theta = 304°$$

The measure of θ is 304°.

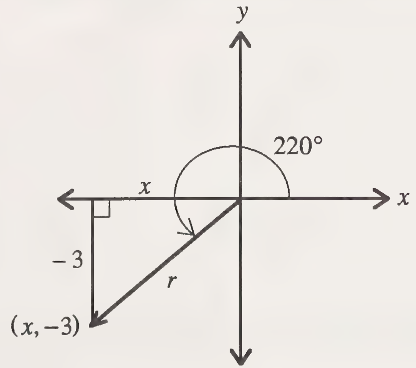
Case 6

You are given y and θ, and you need to find x and r.



Draw a perpendicular line from the endpoint of the terminal arm to the x-axis. This will form the right-angled triangle.

Now use the sine and tangent functions to find x and r.



$$\sin 220° = \frac{-3}{r}$$
$$r = \frac{-3}{\sin 220°}$$
$$r \doteq \frac{-3}{-0.6428}$$
$$r \doteq 4.7$$

$$\tan 220° = \frac{-3}{x}$$
$$x = \frac{-3}{\tan 220°}$$
$$x \doteq \frac{-3}{0.8391}$$
$$x \doteq -3.6$$

Rounded to one decimal place, the value for r is 4.7 and the value for x is -3.6.

Often the information in a question is not as clear as in the previous cases. Even though the information may not be shown clearly, you should still be able to interpret it correctly to solve the problem. Study the following example.

Example 19

Given that $\cos\ \theta = -\frac{1}{4}$ and that $\tan\ \theta$ is positive, solve for θ and r.

Solution:

The first step is to decide in which quadrant you will find the terminal arm.

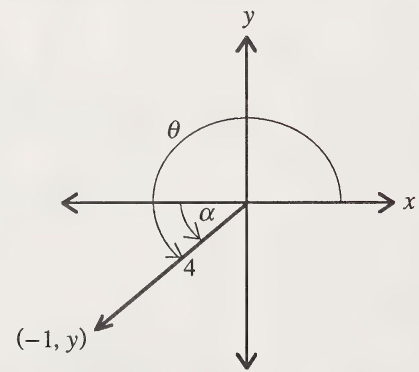
You are told that cosine is negative and tangent is positive. This only happens in Quadrant III.

In Quadrant III both x and y are negative while r is positive. (r is always positive.)

Look at the cosine function. By definition, the cosine ratio is x divided by r. Since the ratio is -1 divided by 4, you can assume that x is -1 and $r = 4$.

To find θ, solve $\cos\ \theta = -\frac{1}{4}$. First solve $\cos\ \alpha = \frac{1}{4}$ to find the reference angle α.

$$\cos\ \alpha = \frac{1}{4}$$
$$= 0.25$$
$$\alpha = 76°$$



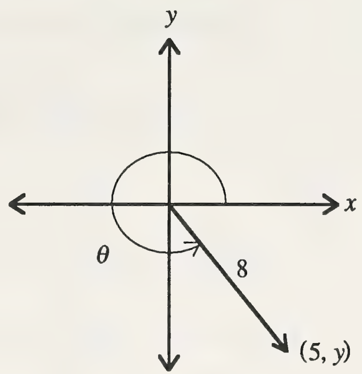
Since you are dealing with an angle in Quadrant III, use the following relationship.

$$\theta = 180° + \alpha$$
$$\theta = 180° + 76°$$
$$\theta = 256°$$

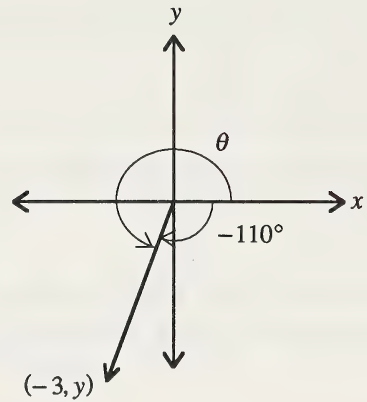
The measure of the angle in standard position is 256°.

Do one of the following questions.

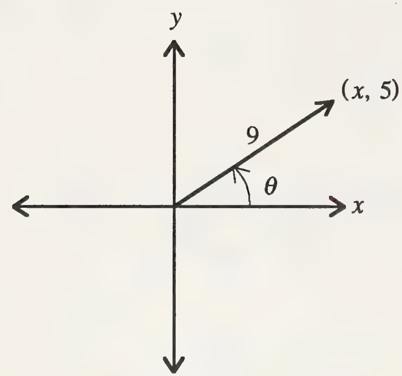
1. a. Find θ and y given the following situation.



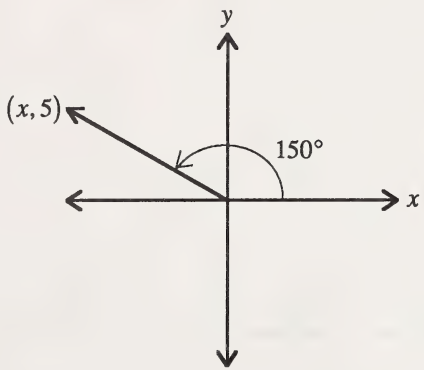
b. Find r and y given the following situation.



c. Find x and θ given the following situation.



d. Find r and x given the following situation.



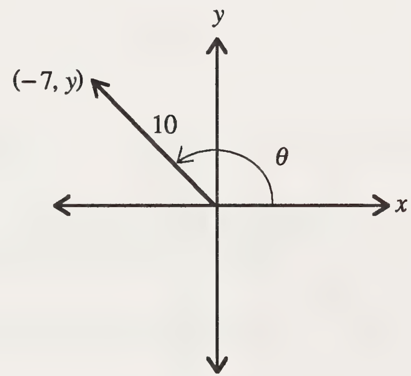
e. Find r and θ given that r terminates at $(-5, -3)$.

f. Find x and y given that $\theta = 144°$ and $r = 15$.

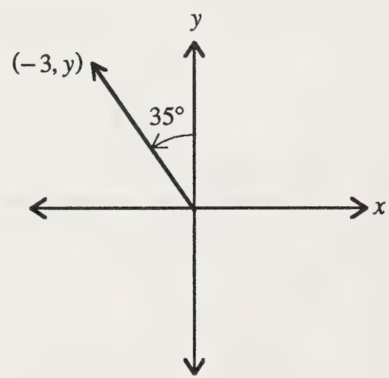
g. Given that $\tan\ \theta = \frac{12}{5}$ and $\sin\ \theta$ is negative, find θ one possible terminal point (x, y), and the corresponding r-value.

h. Given that $\sin\ \theta = \frac{-2}{3}$ and θ is an angle in the fourth quadrant, find θ, one possible terminal point (x, y), and the corresponding r-value.

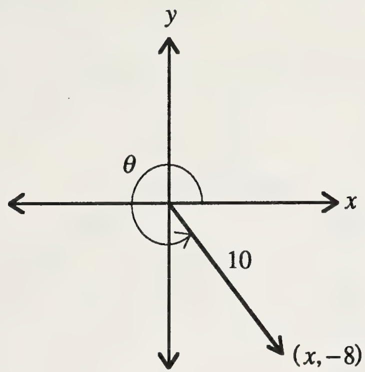
2. a. Find y and θ given the following situation.



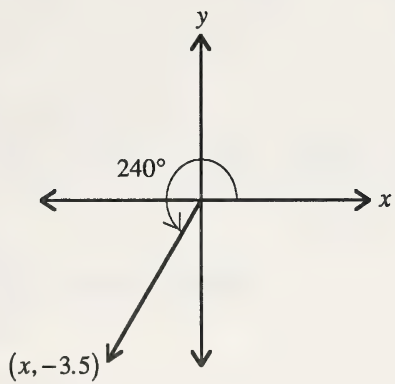
b. Find y and r given the following situation.



c. Find θ and x given the following situation.



d. Find x and r given the following situation.



e. Find θ and r given that r terminates at $(4, -7)$.

f. Find y and x given that $r = 12$ and $\theta = -220°$.

g. Given that $\tan\ \theta = \frac{-12}{5}$ and $\cos\ \theta$ is negative, find θ, one possible terminal point (x, y), and the corresponding r-value.

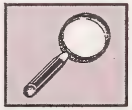
h. Given that $\cos\ \theta = \frac{-3}{4}$ and θ is an angle in the second quadrant, find θ, one possible terminal point (x, y), and the corresponding r-value.

For solutions to **Activity 4**, turn to **Appendix A, Topic 2**.

If you require help, do the Extra Help section.

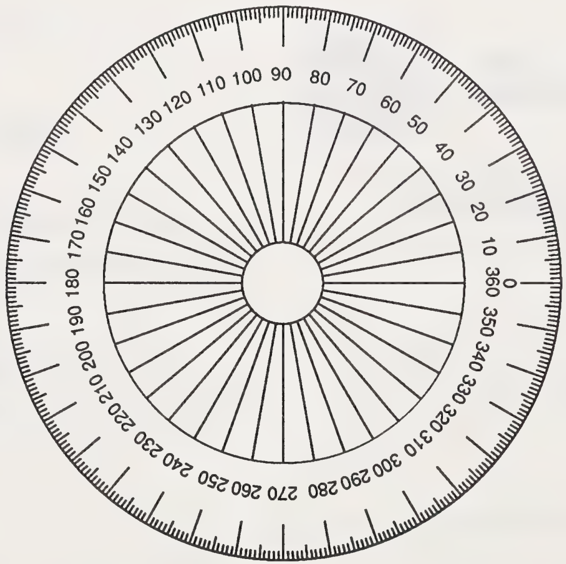
If you want more challenging explorations, do the Extensions section.

} You may decide to do both.

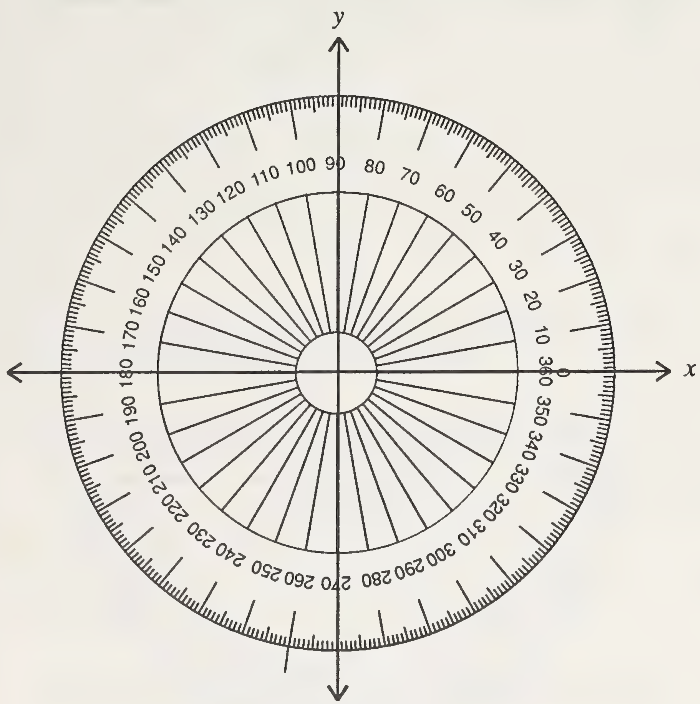


Extra Help

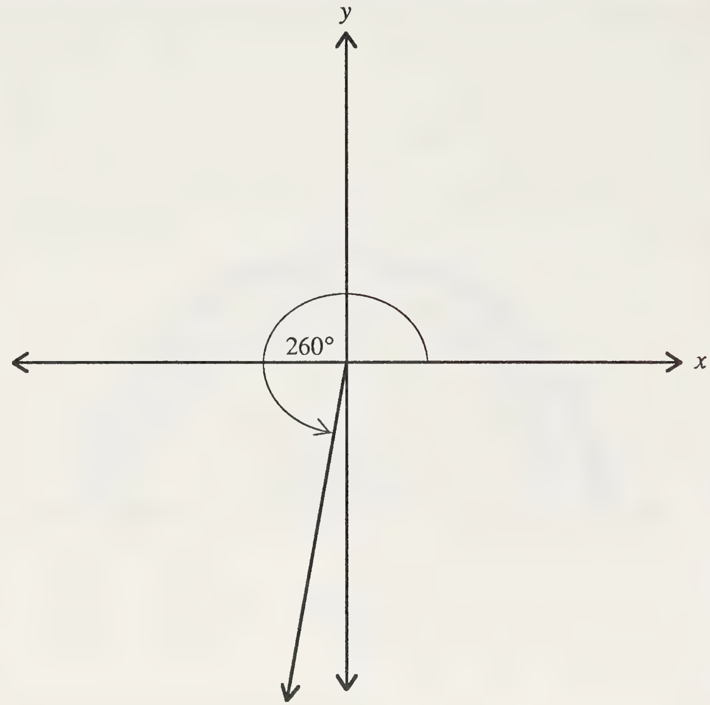
Sometimes students have problems measuring angles greater than 180°. The reason is probably due to the protractor. The protractor is used to measure angles directly up to 180°. Angles greater than 180° must be measured indirectly. Here is a picture of a circular protractor with only one scale, from 0° to 360°.



If you had a circular protractor you could measure any angle between 0° and 360° directly. For example, draw an angle of 260° using the circular protractor.

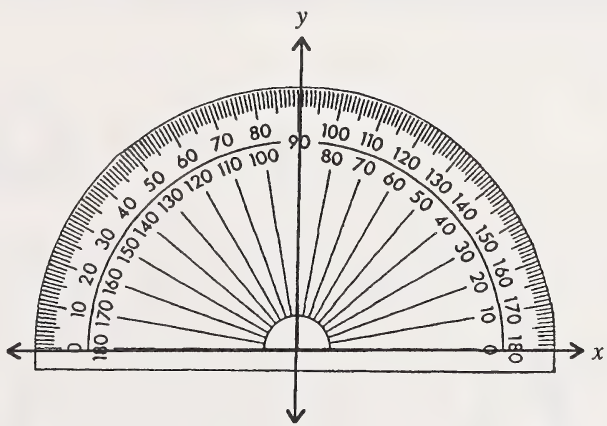


Remove the protractor and an angle appears as shown.

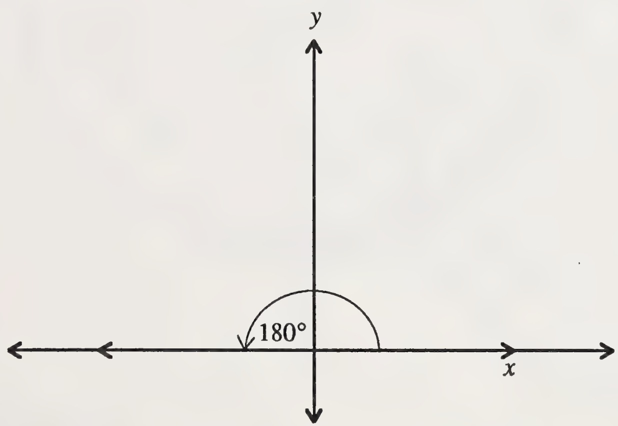


How is this to be done with only a semicircular protractor? It can be done when you consider that **a straight line measures 180°**.

Draw a 180° angle on the coordinate plane using a semicircular protractor.



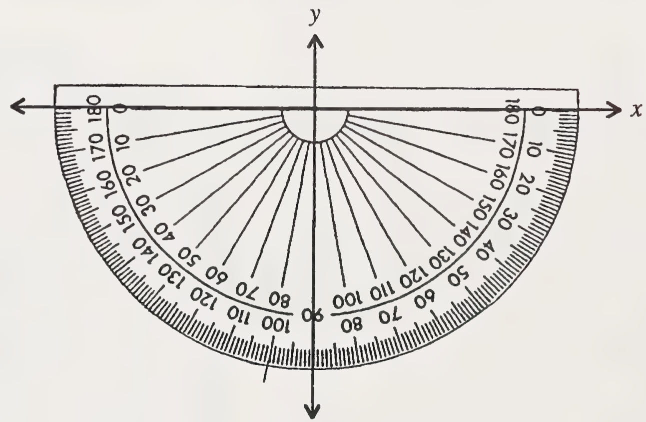
Remove the protractor and draw the angle.

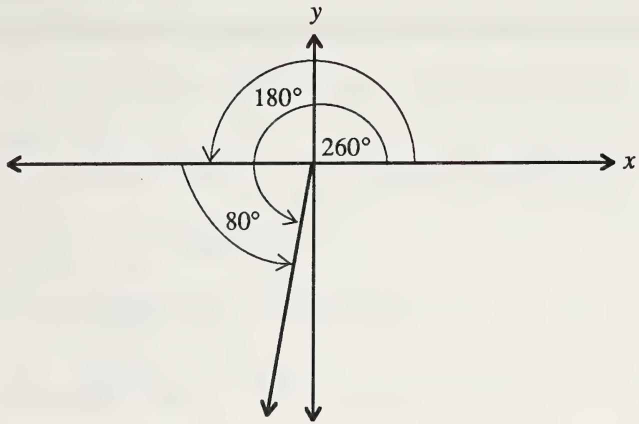


A 180° angle has the initial arm as the positive x-axis and the terminal arm as the negative x-axis. Therefore, any angle greater than 180° and less than or equal to 360° is measured from the negative x-axis by using the difference of the angle and 180°.

For example, draw a 260° angle on the coordinate plane.

Since 260° is greater than 180° and is less than 360°, you need to draw the difference of 260° and 180° (260° – 180° = 80°) measuring from the negative x-axis.





Try some of these questions to ensure that you understand how to draw and measure angles between 180° and 360°.

1. Calculate the angle you will have to draw from the negative x-axis in order to draw the required angle.

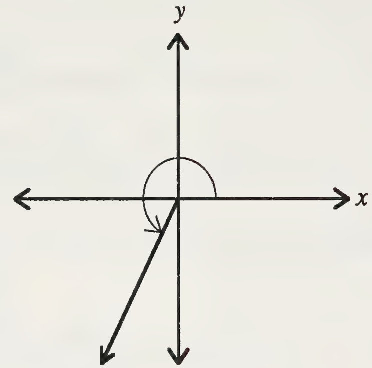
a. 345°	b. 195°
c. 270°	d. 290°
e. 210°	f. 355°
g. 175°	h. 360°

2. Draw the following angles on a coordinate plane in standard position. Graph paper is provided in **Appendix B**.

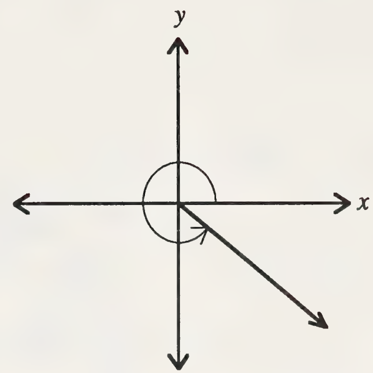
a. 275°	b. 310°
c. 190°	d. 340°

3. Measure the following angles.

a.



b.



For solutions to **Extra Help**, turn to **Appendix A, Topic 2**.

Extensions

In this topic the degree has been the unit used for measuring angles. There is another unit that you can use to measure angles. This unit is called a radian. If the radius of a circle is r units and the length of an arc is also r units, then the measure of the central angle subtended by this arc is 1 radian. According to this definition, the measure of any angle θ is as follows:

$$\theta(\text{in radians}) = \frac{\text{arc length of } \theta}{\text{radius } r} = \frac{a}{r}$$

For a circle of radius $1(r=1)$, the radian measure of an angle equals the length of the arc.

In one complete revolution, the circumference of a circle is $2\pi r$. Therefore, the radian measure of a 360° angle is $\frac{2\pi r}{r} = 2\pi$ radians.

In other words, $360° = 2\pi$ radians or $180° = \pi$ radians.

If $180° = \pi$ radians, then $1° = \frac{\pi}{180°}$ radians

and $1 \text{ radian} = \frac{180°}{\pi}$

These two formulas can help you to convert from one measure to another.

Example 20

Express each angle in radians. Express answers as a fraction of π.

- 24°

Solution:

$$\begin{aligned} 24° &= 24° \times \frac{\pi}{180°} \\ &= \frac{24°}{180°}\pi \\ &= \frac{2}{15}\pi \text{ radians} \end{aligned}$$

- −52°

Solution:

$$\begin{aligned} -52° &= -52° \times \frac{\pi}{180°} \\ &= \frac{-13}{45}\pi \text{ radians} \end{aligned}$$

- 232°

Solution:

$$\begin{aligned} 232° &= 232° \times \frac{\pi}{180°} \\ &= 1\frac{13}{45}\pi \text{ radians} \end{aligned}$$

Example 21

Express each angle in degrees. Express your answer to two decimal places.

- $\frac{2}{9}\pi$

Solution:

$$\begin{aligned}\frac{2}{9}\pi &= \frac{2}{9}\pi \times \frac{180^\circ}{\pi}\\ &= 40.00^\circ\end{aligned}$$

- 32π

Solution:

$$\begin{aligned}32\pi &= 32\pi \times \frac{180^\circ}{\pi}\\ &= 5760.00^\circ\end{aligned}$$

- $-\frac{5}{7}\pi$

Solution:

$$\begin{aligned}-\frac{5}{7}\pi &= -\frac{5}{7}\pi \times \frac{180^\circ}{\pi}\\ &= -128.57^\circ\end{aligned}$$

If an angle is given in radian measure, how do you use a calculator to find the basic trigonometric functions? You have to follow these steps:

Step 1: Put your calculator in radian mode.

Step 2: Enter the radian measure such as $\frac{2}{5}\pi$ using the π key. The radian measure will be converted to a decimal numeral.

Step 3: Press the function key (sine, cosine, or tangent).

Example 22

Use a calculator to calculate the following.

- $\sin\left(\frac{5\pi}{3}\right)$

Solution:

Enter	Display
$\boxed{\text{Mode}}$ "Rad"	0
$\boxed{\text{Inv}}$ π	3.141592654
$\boxed{\times}$ 5 $\boxed{\div}$ 3 $\boxed{=}$	5.235987756
$\boxed{\sin}$	−0.866025403

$$\therefore \sin\left(\frac{5\pi}{3}\right) = -.866\,025\,403$$

- $\cos\left(-\frac{\pi}{2}\right)$

Solution:

Enter	Display
Mode "Rad"	0
Inv π ÷ 2 =	1.570796327
+/-	−1.570796327
cos	0

$\therefore \cos\left(-\frac{\pi}{2}\right) = 0$

- $\tan\left(\frac{2}{3}\pi\right)$

Solution:

Enter	Display
Mode "Rad"	0
Inv π × 2 ÷ 3 =	2.094395102
tan	−1.732050808

$\therefore \tan\left(\frac{2\pi}{3}\right) \doteq -1.732\ 050\ 808$

Do questions 1 to 3 or 4 to 6.

1. Express each angle in radians. Express your answers as a fraction of π.

   a. $35°$

   b. $-120°$

   c. $320°$

2. Express each angle in degrees.

   a. 14π

   b. $-\frac{2}{5}\pi$

   c. $\frac{4}{5}\pi$

3. Use a calculator to determine the following.

   a. $\sin\left(\frac{3\pi}{2}\right)$

   b. $\cos(3\pi)$

   c. $\tan\left(-\frac{\pi}{4}\right)$

4. Express each angle in radians. Express your answers as a fraction of π.

   a. $-42°$ b. $230°$ c. $400°$

5. Express each angle in degrees. Express your answer to two decimal places in question 5c.

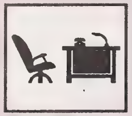
   a. 11π b. $\frac{5}{3}\pi$ c. $-\frac{3}{7}\pi$

6. Use a calculator to determine the following.

   a. $\sin\left(-\frac{7\pi}{6}\right)$ b. $\cos\left(\frac{5\pi}{4}\right)$ c. $\tan\left(\frac{11\pi}{6}\right)$

For solutions to **Extensions**, turn to **Appendix A, Topic 2**.

Topic 3 Sine and Cosine Laws



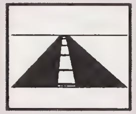
Introduction

All of the triangles you have studied so far have been right-angled triangles.

Are there trigonometric ratios for triangles that do not have a right angle in them? The answer is yes!

Only two laws are required to solve any triangle using trigonometry.

In this topic you will learn the sine and cosine laws. You will also learn how to use them when solving problems involving triangles.



What Lies Ahead

Throughout this topic you will learn to

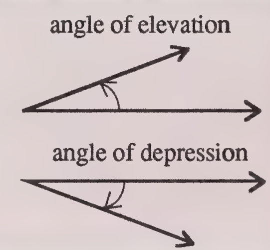
1. find the measures of sides and angles in multiple right triangles and solve problems involving multiple right triangles in two or three dimensions
2. find the measures of unknown sides and angles in oblique triangles by applying the sine law
3. find the measures of unknown sides and angles in oblique triangles by applying the cosine law

Now that you know what to expect, turn the page to begin your study of sine and cosine laws.

Exploring Topic 3

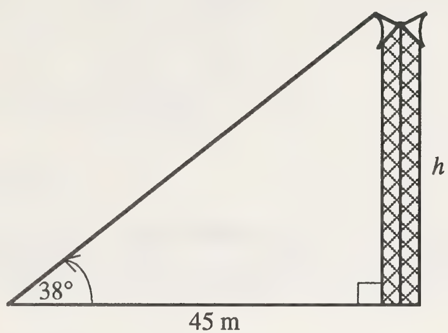
Activity 1

Find the measures of sides and angles in multiple right triangles and solve problems involving mutiple right triangles in two or three dimensions.



You can often make measurements and calculations involving a single right-angled triangle.

For example, you can determine the height of a microwave tower as follows. Measure a horizontal distance from the tower. Find the angle of elevation to the top of the tower. Calculate the height using trigonometry.



$$\tan 38° = \frac{h}{45 \text{ m}}$$

$$h = (45 \text{ m})(\tan 38°)$$

$$h \doteq (45 \text{ m})(0.781\,285\,626)$$

$$h \doteq 35 \text{ m}$$

The tower has a height of about 35 m.

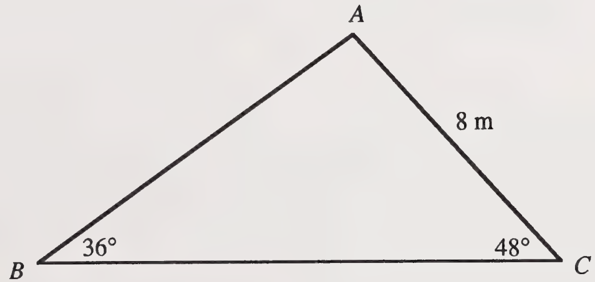
Sometimes you cannot calculate a required distance directly, so you must use multiple right triangles.

Example 1

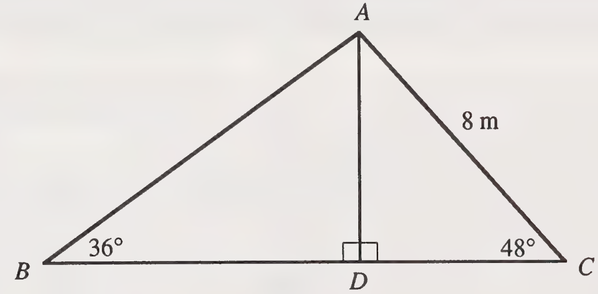
An engineer is required to make roof trusses with one angle of 36° and the other of 48°. The length of the side (AC) adjacent to the 48° angle is 8 m. How wide is the span of the trusses? (The span is BC.)

Solution:

Make a diagram.



Draw a perpendicular line to the base of the truss from point A.



Calculate the length of sides AD and DC.

$$\sin 48° = \frac{AD}{8\text{ m}}$$

$$AD = (8\text{ m})(\sin 48°)$$

$$AD \doteq 5.945\,158\,604\text{ m}$$

$$AD \doteq 5.95\text{ m}$$

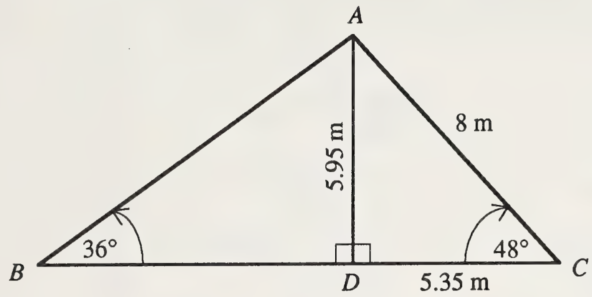
$$\cos 48° = \frac{DC}{8\text{ m}}$$

$$DC = (8\text{ m})(\cos 48°)$$

$$DC \doteq 5.353\,044\,851\text{ m}$$

$$DC \doteq 5.35\text{ m}$$

Now you have enough information to find side BD.



$$\tan 36° \doteq \frac{5.95\text{ m}}{BD}$$

$$BD \doteq \frac{5.95\text{ m}}{\tan 36°}$$

$$BD \doteq \frac{5.95\text{ m}}{0.726\,542\,528}$$

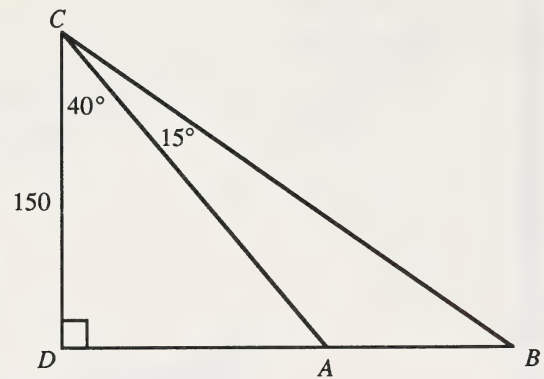
$$BD \doteq 8.19\text{ m}$$

The span of the trusses is approximately equal to $8.19\text{ m} + 5.35\text{ m} \doteq 13.54\text{ m}$.

Example 2

Determine the distance from A to B in the following diagram.

Solution:



$$\tan 40° = \frac{DA}{150}$$

$$DA = 150(\tan 40°) \doteq 125.86$$

$$\tan 55° = \frac{DB}{150}$$

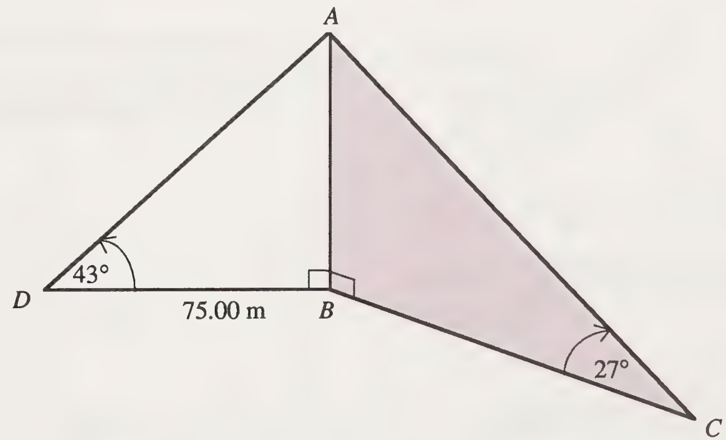
$$DB = 150(\tan 55°) \doteq 214.22$$

$$AB = DB - DA \doteq 214.22 - 125.86 \doteq 88.36$$

Point A is 88.36 units from point B.

Example 3

Two right-angled triangles have side AB in common. The triangles are in three-dimensional space. Determine the lengths of AD and AC.



Solution:

$$\cos 43° = \frac{75.00 \text{ m}}{AD}$$

$$AD = \frac{75.00 \text{ m}}{\cos 43°}$$

$$AD \doteq \frac{75.00 \text{ m}}{0.731\,353\,701}$$

$$AD \doteq 102.55 \text{ m}$$

$$\tan 43° = \frac{AB}{75.00 \text{ m}}$$

$$AB = (75.00 \text{ m})(\tan 43°)$$

$$AB \doteq 69.938\,631\,46 \text{ m}$$

$$AB \doteq 69.94 \text{ m}$$

$$\sin 27° = \frac{AB}{AC}$$

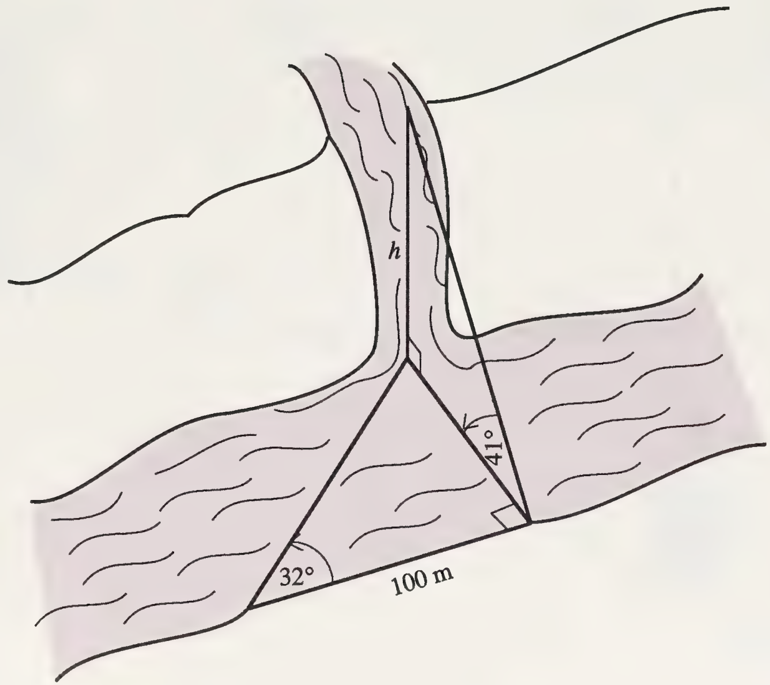
$$AC = \frac{AB}{\sin 27°}$$

$$AC \doteq \frac{69.94 \text{ m}}{0.453\,990\,499}$$

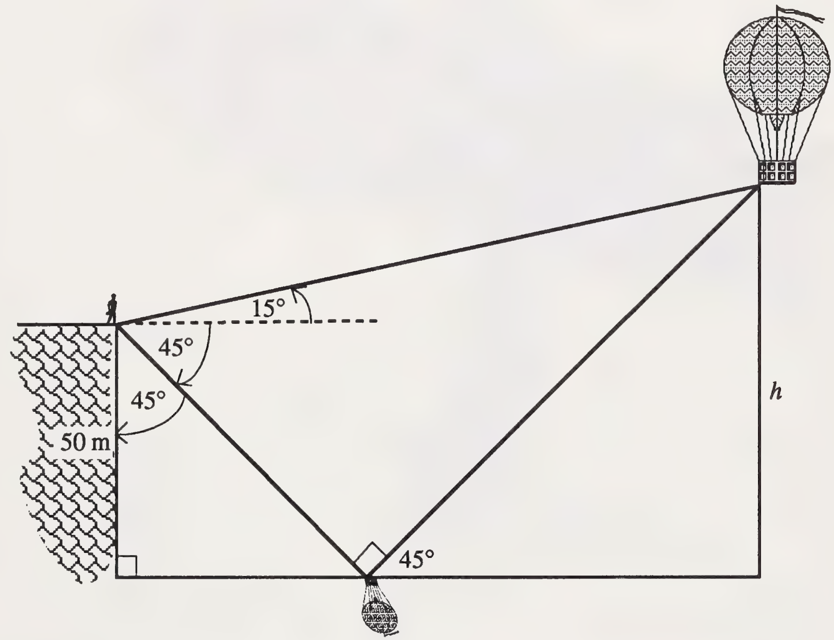
$$AC \doteq 154.06 \text{ m}$$

Do questions 1 and 2 and any four of questions 3 to 8.

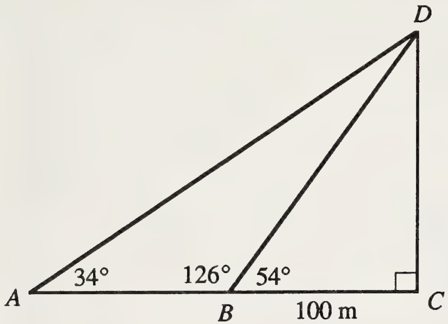
1. Lava from a volcano is flowing down a cliff as shown in the diagram. A scientist wishes to measure the height of the cliff and the width of the lava flow at ground level. The scientist is able to make the measurements shown in the diagram. Calculate the height of the cliff and the width of the lava flow at ground level.



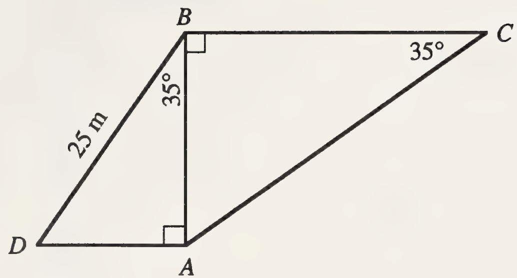
2. A hiker climbs to the top of a cliff which is 50 m high and overlooks a smooth lake. The hiker sees a balloon above the lake and a reflection of the balloon in the lake. The hiker quickly measures the angle of elevation to the balloon to be 15° and the angle of depression to the reflection of the balloon in the lake to be 45°. How high is the balloon above the lake? How high is the balloon above the hiker?



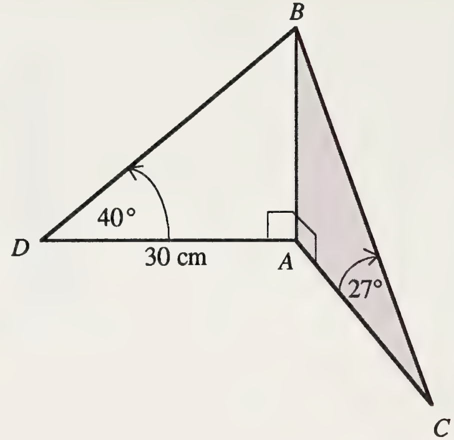
3. Calculate AD given the following diagram.



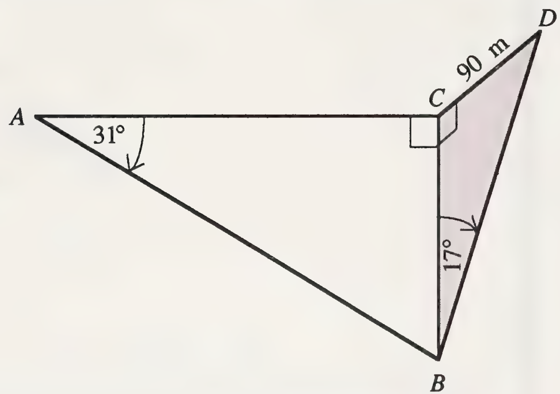
4. Calculate AC given the following diagram.



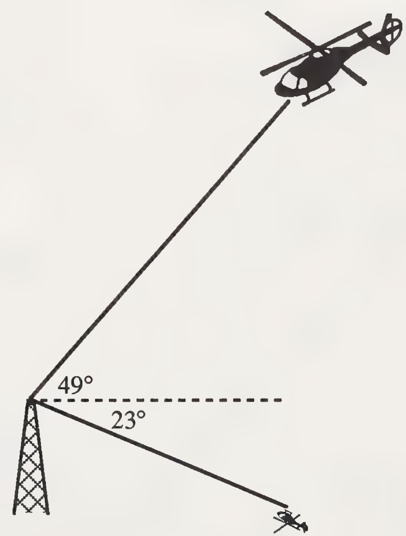
5. Calculate BC given the following diagram.



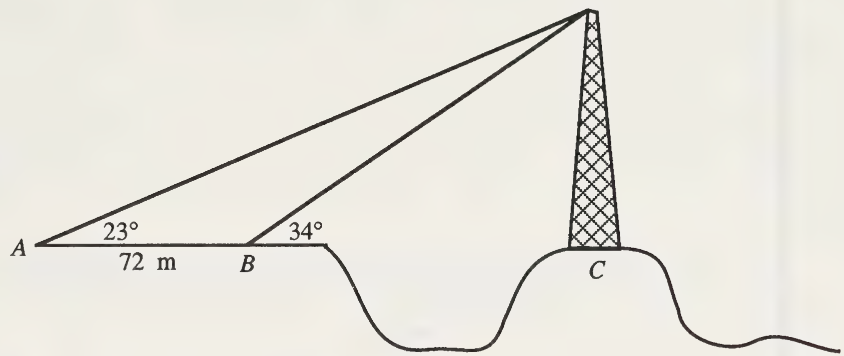
6. Calculate AB given the following diagram.



7. The sun is positioned directly overhead. A helicopter is hovering some distance from a fire tower that is 36.00 m high. An observer at the top of the tower measures the angle of elevation to the helicopter to be 49° and the angle of depression to the shadow of the helicopter to be 23°. How far is the shadow of the helicopter from the base of the tower? How high is the helicopter above its shadow?



8. A tower is supported by two guy wires as shown in the diagram. Calculate the height of the tower and the length of each guy wire. (Note: This is a very complex question.)



For solutions to **Activity 1**, turn to **Appendix A, Topic 3**.

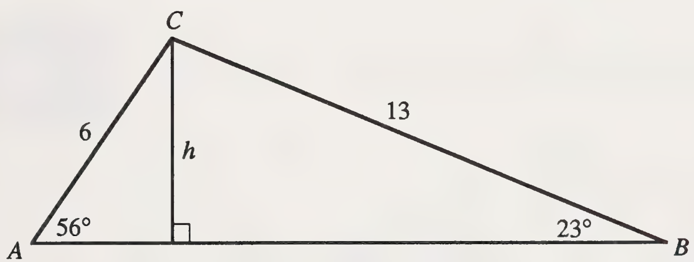
Activity 2

Find the measures of unknown sides and angles in oblique triangles by applying the sine law.

In previous work you have learned how to apply trigonometric ratios when solving right triangles. You have also learned to calculate the values of trigonometric ratios for angles whose measures are greater than 90° or less than 0°. Is it possible to solve oblique triangles using trigonometric ratios?

Oblique triangles do not contain a 90° angle.

Consider the following acute triangle.



$$\sin 56° = \frac{h}{6} \quad \text{and} \quad \sin 23° = \frac{h}{13}$$

$$h = 6 \sin 56° \qquad\qquad h = 13 \sin 23°$$

Since h will equal itself, $13 \sin 23° = 6 \sin 56°$.

Divide both sides of the equation by $\sin 23°$ and $\sin 56°$.

$$\frac{13 \sin 23°}{\sin 23° \sin 56°} = \frac{6 \sin 56°}{\sin 23° \sin 56°}$$

$$\frac{13 \cancel{\sin 23°}^{1}}{\cancel{\sin 23°}_{1} \sin 56°} = \frac{6 \cancel{\sin 56°}^{1}}{\sin 23° \cancel{\sin 56°}_{1}}$$

$$\frac{13}{\sin 56°} = \frac{6}{\sin 23°}$$

Looking back at the diagram, you will note that side a has a length of 13 units, $\angle A$ has a measure of 56°, b has a length of 6 units, and $\angle B$ has a measure of 23°. Put this information into the equation.

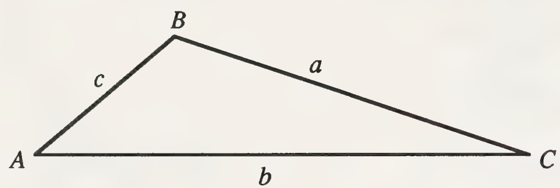
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

A similar relationship can be found for side c and $\angle C$.

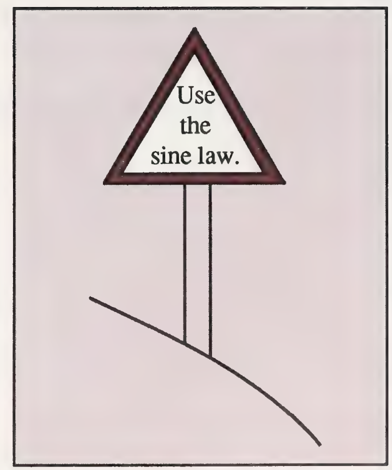
This relationship is called the **sine law**.

For any triangle ABC with sides a, b, and c, the following is true:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

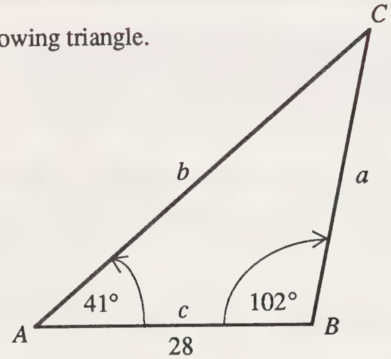


The **Extensions** section of this topic contains a proof of the sine law for oblique triangles.



Example 4

Solve the following triangle.



Solution:

Solving the triangle means finding the measures of all sides and all angles which are not provided.

The sum of the interior angles of any triangle is 180°.

$$\text{Therefore, } \angle C = 180° - (41° + 102°)$$
$$= 37°$$

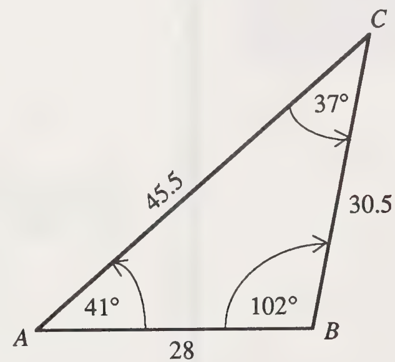
To find side a, do the following:

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$
$$a = \frac{c}{\sin C}(\sin A)$$
$$a = \frac{28}{\sin 37°}(\sin 41°)$$
$$a \doteq \frac{28}{(0.6018)}(0.6561)$$
$$a \doteq 30.5$$

To find side b, do the following:

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$
$$b = \frac{c}{\sin C}(\sin B)$$
$$b = \frac{28}{\sin 37°}(\sin 102°)$$
$$b \doteq \frac{28}{(0.6018)}(0.9781)$$
$$b \doteq 45.5$$

A summary of the solved triangle is shown as follows.



There are a variety of ways you can choose to solve the triangle including the method used in Example 4.

For example, you could have solved for b first by using the following.

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

Then you could have solved for a using the following.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

If you use your calculator to find sin 102°, you will not have any difficulty. If you use the trigonometric table, you must first find the reference angle.

$$\begin{aligned}\text{Reference angle} &= 180° - 102° \\ &= 78°\end{aligned}$$

Then you must determine the sine of 78° from the table.

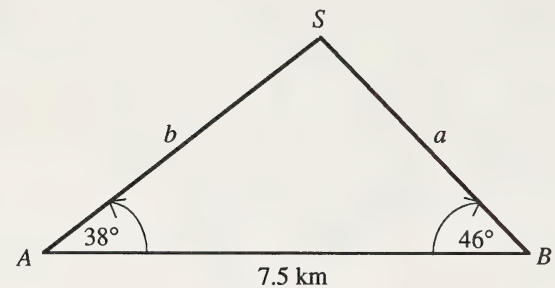
$$\sin 78° \doteq 0.9781$$

Since 102° is in Quadrant II, the sign is positive.

In fact, no interior angle of a triangle can be greater than 180°. Therefore, the sine function will always be positive when solving triangles.

Example 5

Two lighthouses are separated by a distance of 7.5 km. A ship passes some distance from the lighthouses. Measurements were made from each lighthouse at the same time. Lighthouse A measured the angle to the ship to be 38° and lighthouse B measured the angle to the ship to be 46° as shown in the following diagram. Find the distance of the ship from each lighthouse.



Solution:

To find the third angle, do the following:

$$\begin{aligned}\angle S &= 180° - (38° + 46°) \\ &= 96°\end{aligned}$$

To find distance a, do the following:

$$\begin{aligned}\frac{a}{\sin A} &= \frac{7.5 \text{ km}}{\sin S} \\ a &= \frac{7.5 \text{ km}}{\sin S}(\sin A) \\ a &= \frac{7.5 \text{ km}}{\sin 96°}(\sin 38°) \\ a &\doteq \frac{7.5 \text{ km}}{(0.9945)}(0.6157) \\ a &\doteq 4.64 \text{ km}\end{aligned}$$

To find distance b, do the following:

$$\frac{b}{\sin B} = \frac{7.5 \text{ km}}{\sin S}$$

$$b = \frac{7.5 \text{ km}}{\sin S}(\sin B)$$

$$b = \frac{7.5 \text{ km}}{\sin 96^\circ}(\sin 46^\circ)$$

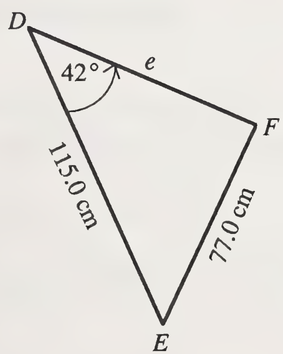
$$b \doteq \frac{7.5 \text{ km}}{(0.9445)}(0.7193)$$

$$b \doteq 5.42 \text{ km}$$

Therefore, the distance from lighthouse A to the ship is approximately 5.42 km and the distance from lighthouse B to the ship is approximately 4.64 km.

Example 6

Solve the triangle.



Solution:

To find $\angle F$, do the following:

$$\frac{f}{\sin F} = \frac{d}{\sin D}$$

$$\sin F = \frac{f \sin D}{d}$$

$$\sin F = \frac{(115.0 \text{ cm})(\sin 42^\circ)}{77.0 \text{ cm}}$$

$$\sin F \doteq \frac{(115.0 \text{ cm})(0.6691)}{77.0 \text{ cm}}$$

$$\sin F \doteq 0.9994$$

$$\angle F \doteq 88^\circ$$

$$\angle E = 180^\circ - (88^\circ + 42^\circ)$$

$$= 50^\circ$$

$$\frac{e}{\sin E} = \frac{d}{\sin D}$$

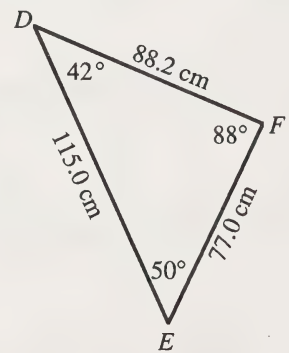
$$e = \frac{d \sin E}{\sin D}$$

$$e = \frac{(77.0 \text{ cm})(\sin 50^\circ)}{\sin 42^\circ}$$

$$e \doteq \frac{(77.0 \text{ cm})(0.7660)}{(0.6691)}$$

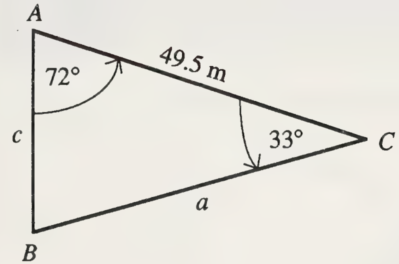
$$e \doteq 88.2 \text{ cm}$$

A summary of the solved triangle is shown as follows.

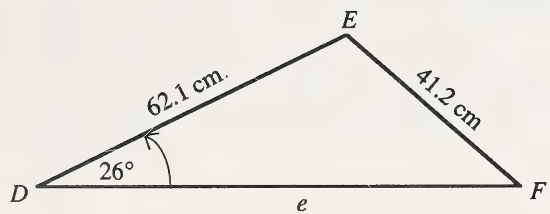


Try any three of the following questions.

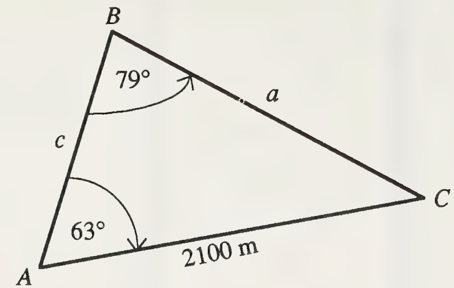
1. Solve the triangle.



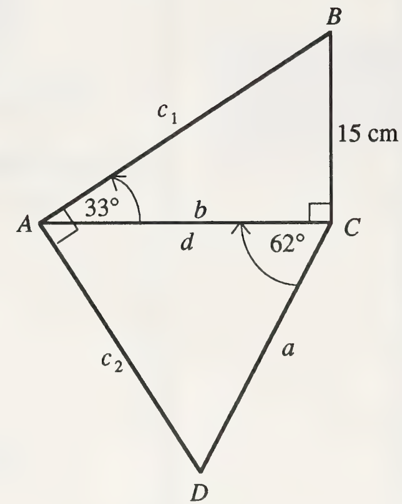
2. Solve the triangle.



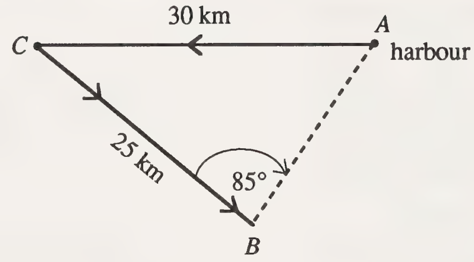
3. Solve the triangle.



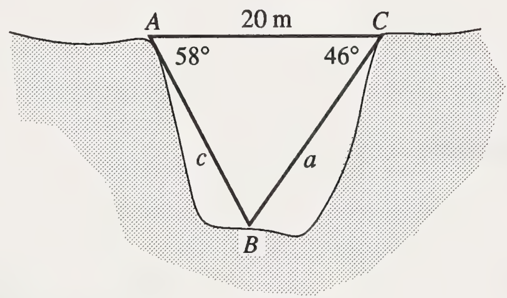
4. Solve both triangles.



5. A ship travels as shown in the diagram. How far is the ship from harbour when it is at point B?



6. Determine the lengths of the supports for the bridge in the accompanying diagram.

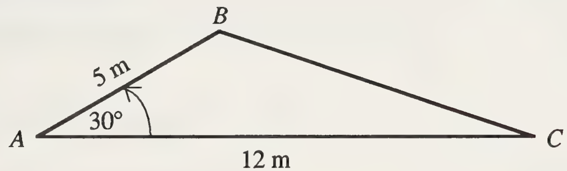


For solutions to **Activity 2**, turn to **Appendix A, Topic 3**.

Activity 3

Find the measures of unknown sides and angles in oblique triangles by applying the cosine law.

What happens when you try to solve the following triangle using the sine law?

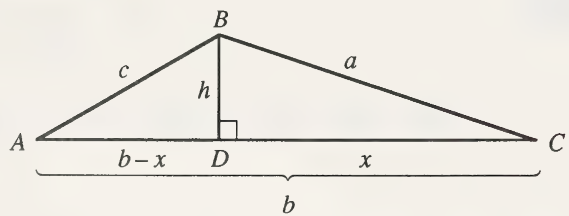


No matter which ratios you choose, you cannot isolate one unknown. There are always two unknowns.

In general you can solve triangles which have SAA, ASA, or SSA as the given information using the sine law. To solve triangles other than these three, you need to use the cosine law. The following shows the relationships of the cosine law.

SAA: side - angle - angle
ASA: angle - side - angle
SSA: side - side - angle

Given the triangle ABC, draw a perpendicular from B to AC, label the measure of AC as b, AD as $b - x$, and label DC as x.



The following information can be derived from right triangle BDC.

$$\cos C = \frac{x}{a} \qquad \text{and} \qquad a^2 = h^2 + x^2$$

$$x = a \cos C$$

The following information can be derived from right triangle ABD.

$$c^2 = h^2 + (b - x)^2$$

$$c^2 = h^2 + b^2 - 2bx + x^2$$

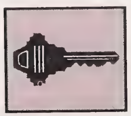
$$c^2 = b^2 + \left(h^2 + x^2\right) - 2bx$$

Substitute the expressions for $h^2 + x^2$ and x into this equation.
$\left(a^2 = h^2 + x^2 \text{ and } x = a \cos C\right)$

$$c^2 = b^2 + a^2 - 2ba \cos C$$

Similar relationships can be found for b^2 and a^2.

This relationship is called the **cosine law**.

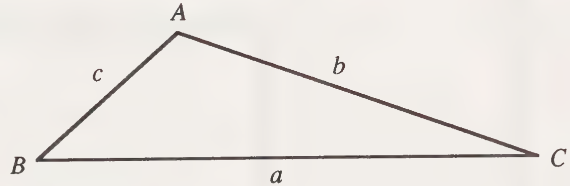


For any triangle ABC with sides a, b, and c, the following applies:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

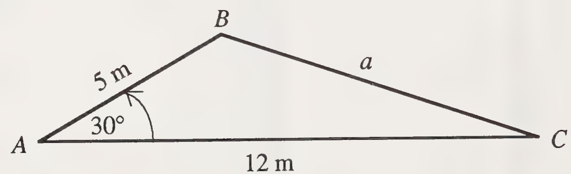


Use your calculator to make sure the following is true.
If an angle is acute (0° < measure of angle < 90°), then cosine will be positive. If an angle is obtuse (90° < measure of angle < 180°), then cosine will be negative. The sign of the cosine ratio will be important in the calculations that involve the equations of the cosine law.

The cosine law will be used to solve the following problems.

Example 7

Solve the triangle.



Solution:

Use the appropriate equation.

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$a^2 = (12)^2 + (5)^2 - 2(12)(5)(\cos 30°)$$
$$a^2 \doteq 144 + 25 - 120(0.8660)$$
$$a^2 \doteq 169 - 103.92$$
$$a^2 \doteq 65$$
$$a \doteq \sqrt{65}$$
$$a \doteq 8$$

The length of side a is about 8 m.

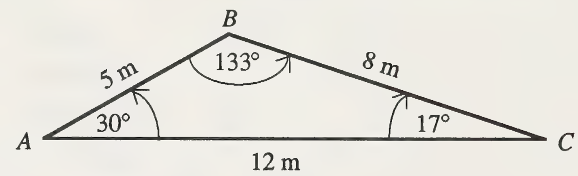
Now use the equations of the cosine law to solve for the other angles.

$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$\cos B = \frac{b^2 - a^2 - c^2}{-2ac}$$
$$\cos B \doteq \frac{(12)^2 - (8)^2 - (5)^2}{-2(8)(5)}$$
$$\cos B \doteq \frac{55}{-80}$$
$$\cos B \doteq -0.6875$$
$$\angle B \doteq 133°$$

If you are using the trigonometric table, you know the angle is obtuse since the cosine is negative. (It must be in Quadrant II.) Therefore, the related angle must be 47° and $\angle B = 180° - 47° = 133°$.

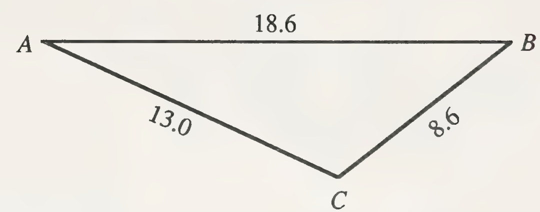
$$\angle C = 180 - (133° + 30°)$$
$$= 17°$$

The triangle is now solved and is shown as follows.



Example 8

Solve the following triangle.



Solution:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\cos A = \frac{a^2 - b^2 - c^2}{-2bc}$$

$$= \frac{(8.6)^2 - (13.0)^2 - (18.6)^2}{-2(13.0)(18.6)}$$

$$\cos A \doteq 0.9119$$

$$\angle A \doteq 24°$$

$$\cos B = \frac{b^2 - a^2 - c^2}{-2ac}$$

$$= \frac{(13.0)^2 - (8.6)^2 - (18.6)^2}{-2(8.6)(18.6)}$$

$$\cos B \doteq 0.7843$$

$$\angle B \doteq 38°$$

$$\angle C \doteq 180° - (38° + 24°)$$

$$\angle C \doteq 118°$$

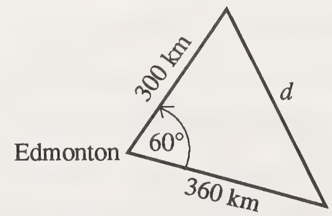
(Answers could vary from these due to rounding.)

Example 9

Two planes leave Edmonton at the same time, flying in directions that make an angle of 60° with each other. If the speeds of the planes are 300 km/h and 360 km/h, how far apart will the planes be after one hour?

Solution:

After one hour the planes will have travelled 300 km and 360 km, respectively. If d represents the distance between the planes after one hour, then the diagram for the problem would be as follows:



$$d^2 = 300^2 + 360^2 - 2(300)(360)\cos 60°$$

$$d^2 = 90\,000 + 129\,600 - 216\,000(0.5)$$

$$d^2 = 111\,600$$

$$d = \sqrt{111\,600}$$

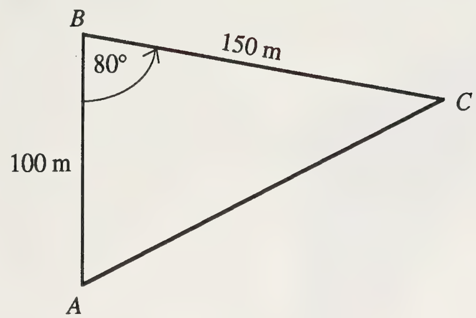
$$d \doteq 334.0658$$

$$d \doteq 334.1$$

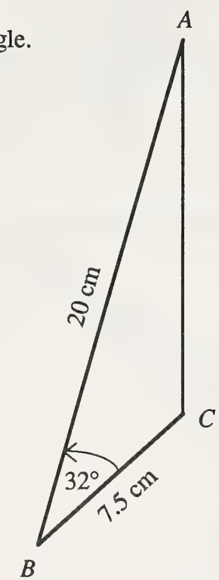
After one hour the planes will be approximately 334.1 km apart.

Try any four of the following.

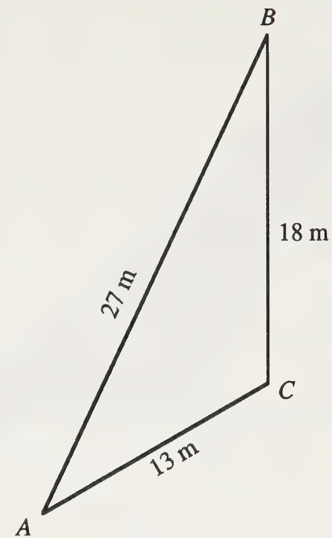
1. Solve the following triangle.



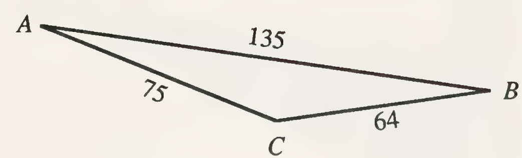
2. Solve the following triangle.



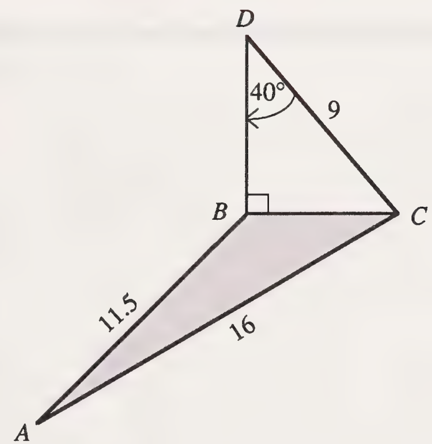
3. Solve the following triangle.



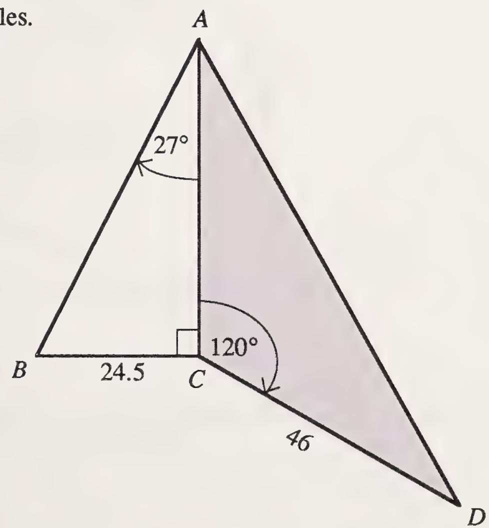
4. Solve the following triangle.



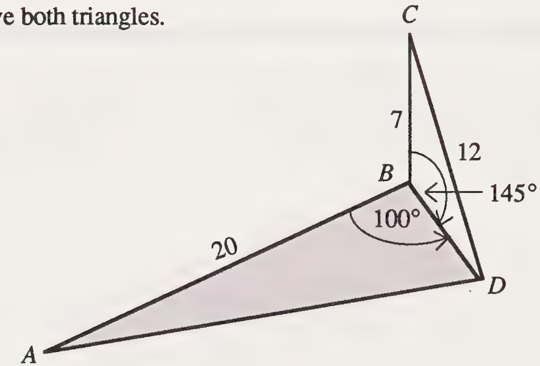
5. Solve both triangles.



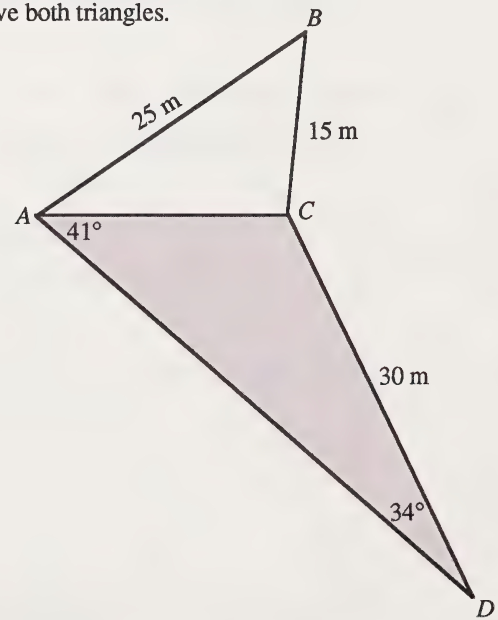
6. Solve both triangles.



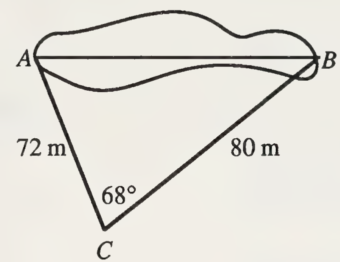
7. Solve both triangles.



8. Solve both triangles.



9. In order to measure the distance across a pond, a surveyor fixes a point C and measures the distances from this point to points A and B at the ends of the pond as shown in the diagram. If the measure of angle ACB is 68°, find the distance AB across the pond.



10. Two ships leave a port at the same time, sailing at 15 km/h and 20 km/h, respectively. Assuming the ships sail in straight lines, how far apart will the ships be after four hours if the angle between their directions is 42°?

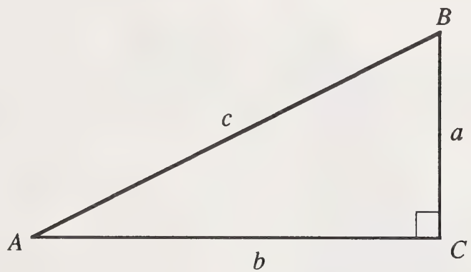
For solutions to **Activity 3**, turn to **Appendix A, Topic 3**.

If you require help, do the Extra Help section.
If you want more challenging explorations, do the Extensions section.
} You may decide to do both.

Extra Help

There are three methods of solving triangles in this unit. There are only two types of triangles that are considered.

- Right-angled triangles: Use the three primary trigonometric ratios and the Pythagorean theorem to solve the triangle.



$\sin A = \frac{a}{c}$ $\sin B = \frac{b}{c}$ $\angle A + \angle B = 90°$

$\cos A = \frac{b}{c}$ $\cos B = \frac{a}{c}$ $c^2 = a^2 + b^2$

$\tan A = \frac{a}{b}$ $\tan B = \frac{b}{a}$

- Triangles other than right-angled triangles: Use the sine and cosine laws.

Only use the sine law for the following situations:

- Two angles and a side opposite one of these two angles are known (SAA).
- Two angles and the included side are known (ASA).
- Two sides and an angle opposite one of these two sides are known (SSA).

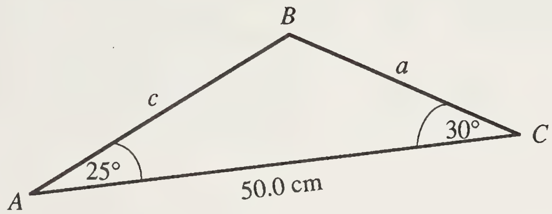
The sine law is as follows:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Select the proportion which will have only one unknown. Solve for the unknown and continue to solve the triangle with this new information until all unknowns are calculated.

Example 10

Solve the following triangle.



Solution:

This is an angle-side-angle situation. Therefore, you can easily calculate the third angle since the sum of the interior angles of a triangle is equal to 180°.

$$\begin{aligned}\angle B &= 180° - (25° + 30°)\\ &= 125°\end{aligned}$$

Complete the solution by using the following:

$$\begin{aligned}\frac{c}{\sin C} &= \frac{b}{\sin B}\\ c &= \frac{b \sin C}{\sin B}\\ c &= \frac{(50.0\text{ cm})(\sin 30°)}{\sin 125°}\\ &\doteq 30.5\text{ cm}\end{aligned}$$

$$\begin{aligned}\frac{a}{\sin A} &= \frac{b}{\sin B}\\ a &= \frac{b \sin A}{\sin B}\\ a &= \frac{(50\text{ cm})(\sin 25°)}{\sin 125°}\\ a &\doteq 25.8\text{ cm}\end{aligned}$$

Only use the cosine law for the following situations:

- Two sides and the included angle are known (SAS).
- Three sides are known (SSS).

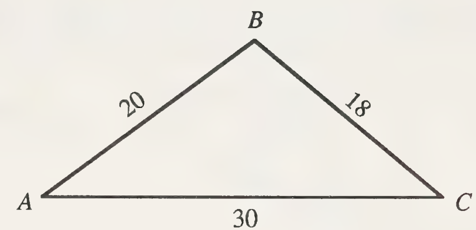
The cosine law is as follows:

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = b^2 + a^2 - 2ab \cos C$$

These are easily memorized by noting the patterns of the letters.

Example 11

Solve the following triangle.



Solution:

Use the cosine law since the SSS situation is given.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\cos A = \frac{a^2 - b^2 - c^2}{-2bc}$$

$$\cos A = \frac{(18)^2 - (30)^2 - (20)^2}{-2(30)(20)}$$

$$\cos A \doteq 0.8133$$

$$\angle A \doteq 36°$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$\cos B = \frac{b^2 - a^2 - c^2}{-2ac}$$

$$\cos B = \frac{(30)^2 - (18)^2 - (20)^2}{-2(18)(20)}$$

$$\cos B \doteq -0.2444$$

$$\angle B \doteq 104°$$

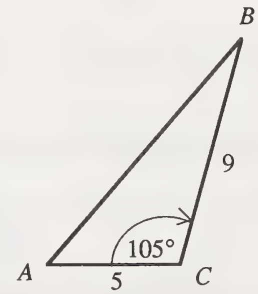
$$\angle C \doteq 180° - (104° + 36°)$$

$$\angle C \doteq 40°$$

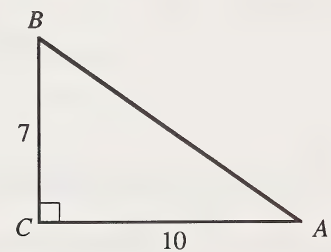
Try any one of the following questions.

1. State which of the three methods you would use to solve each triangle and then solve the triangle.

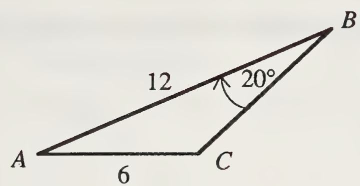
a.



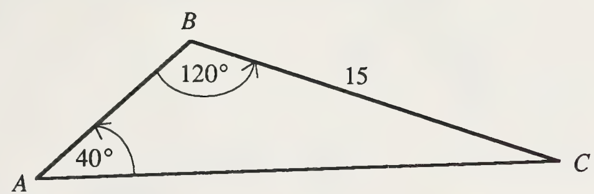
b.



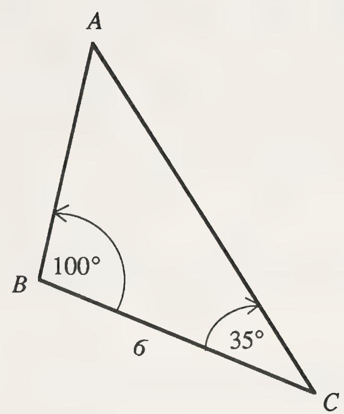
c.



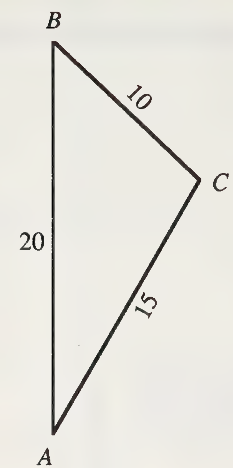
d.



e.

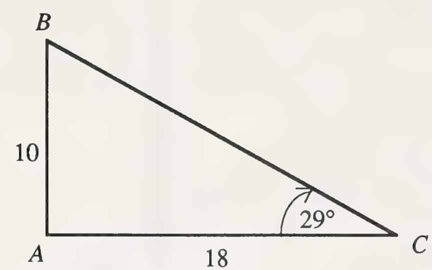


f.

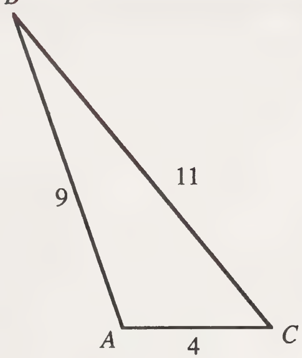


2. State which of the three methods you would use to solve each triangle and then solve the triangle.

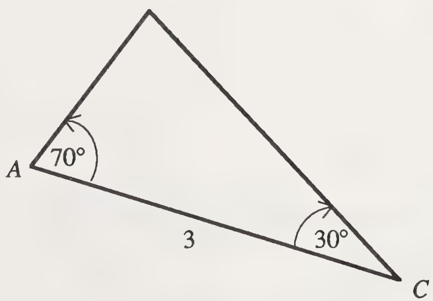
a.



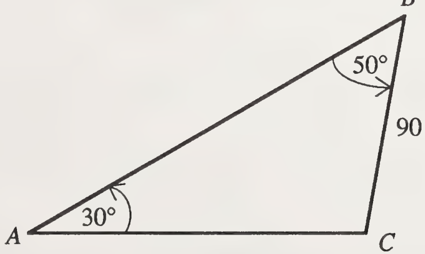
b.



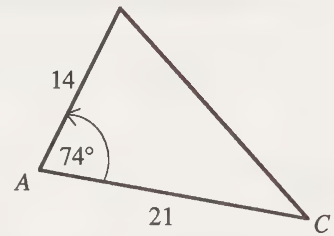
c.



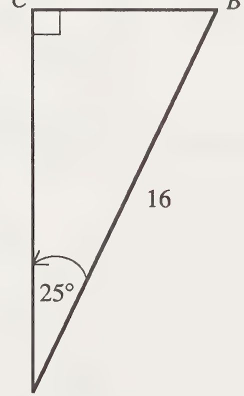
d.



e.



f.

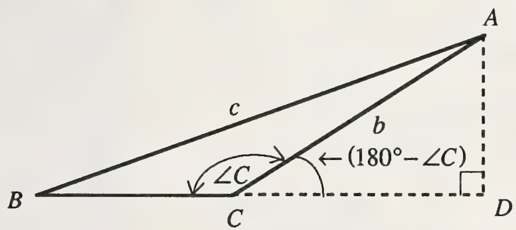


For solutions to **Extra Help**, turn to **Appendix A, Topic 3**.

Extensions

If you have an obtuse-angled triangle, here is how you can prove the sine law.

Given: Triangle ABC is an obtuse-angled triangle. Prove the sine law.
Proof: Draw AD perpendicular to BC extended to D.



In ΔABD,

$$\frac{AD}{c} = \sin B$$

$$\therefore AD = c(\sin B)$$

In ΔACD,

$$\frac{AD}{b} = \sin(180° - \angle C)$$

$$AD = b \times \sin(180° - \angle C)$$

$$\therefore AD = b(\sin C)$$

$\sin(180° - \angle C) = \sin C$

Thus, $AD = c(\sin B) = b(\sin C)$. This statement can be changed to the form $\frac{c}{\sin C} = \frac{b}{\sin B}$.

That is, for any triangle, $\frac{c}{\sin C} = \frac{b}{\sin B}$.

Similarly, by drawing perpendiculars to the other sides, it can be proved that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

Now try the following question.

Use the previous triangle to prove the cosine law for obtuse-angled triangles.

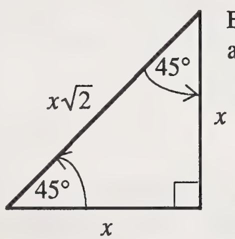
For solutions to **Extensions**, turn to **Appendix A, Topic 3**.

Unit Summary

What You Have Learned

In this unit you have learned the following.

- 45° - 45° - 90° triangles
  - The two shorter sides are the same length.
  - The longer side (hypotenuse) is $\sqrt{2}$ times the length of a side.



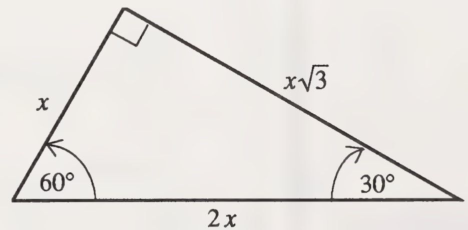
Exact trigonometric ratios are as follows:

$\sin 45° = \frac{\sqrt{2}}{2}$

$\cos 45° = \frac{\sqrt{2}}{2}$

$\tan 45° = 1$

- 30° - 60° - 90° triangles
  - The shorter side is one-half the length of the hypotenuse.
  - The longer side is $\sqrt{3}$ times the length of the shorter side.



Exact trigonometric ratios are as follows:

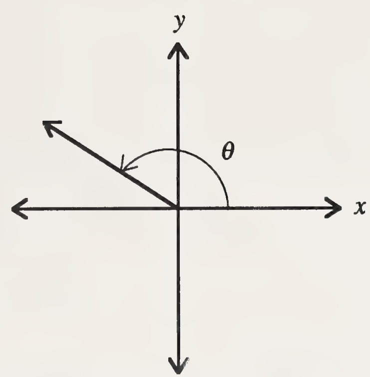
$\sin 30° = \frac{1}{2}$ $\quad$ $\sin 60° = \frac{\sqrt{3}}{2}$

$\cos 30° = \frac{\sqrt{3}}{2}$ $\quad$ $\cos 60° = \frac{1}{2}$

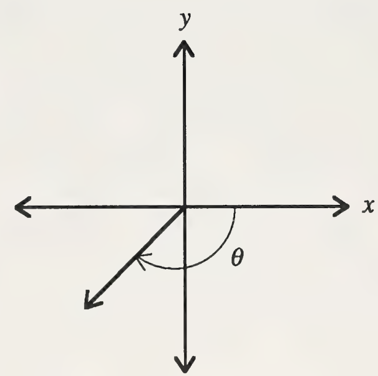
$\tan 30° = \frac{\sqrt{3}}{3}$ $\quad$ $\tan 60° = \sqrt{3}$

Unit Summary

- Positive angles in the coordinate plane are counter-clockwise rotations.



- Negative angles in the coordinate plane are clockwise rotations.



Unit Summary

- The trigonometric functions for angles drawn on the coordinate plane are as follows:

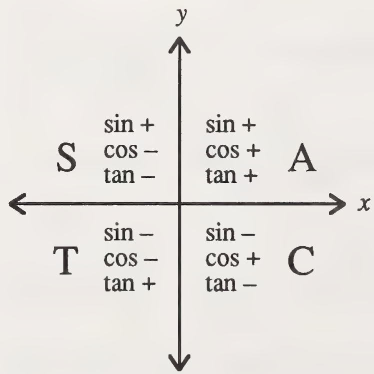
$$\sin\ \theta = \frac{y}{r}$$

$$\cos\ \theta = \frac{x}{r}$$

$$\tan\ \theta = \frac{y}{x}$$

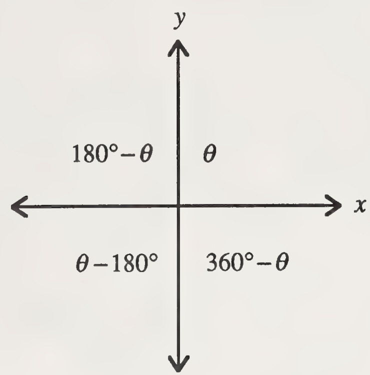
where $r = \sqrt{x^2 + y^2}$

- The signs of the primary trigonometric functions in the coordinate plane can be summarized as in the following chart.



Unit Summary

- The first-quadrant angle which has the same absolute trigonometric values as for a given angle in any one of the other three quadrants is called the reference angle.



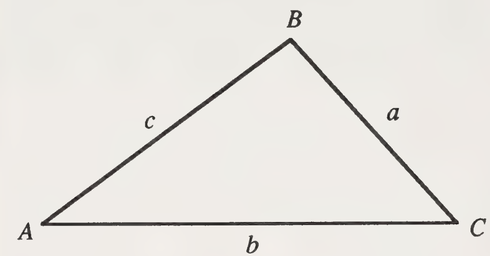
- The reference angle and the trigonometric table can be used to determine the trigonometric ratios for any angle.

- A scientific calculator can be used to calculate the trigonometric ratios for any angle.

- Any two values of x, y, r, and θ can be determined given the other two values.

- The measures of the sides and angles of multiple right triangles can be calculated.

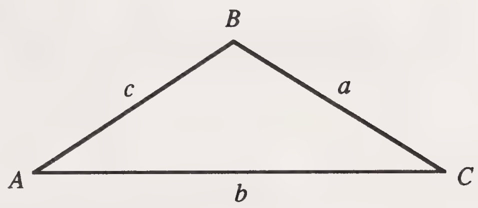
- The sine law for any triangle ABC is as follows:



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Unit Summary

- The cosine law for any triangle ABC is as follows:



$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

You are now ready to complete the **Unit Assignment**.

Appendices

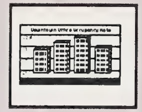
Appendix A
Solutions

Review

Topic 1 45° - 45° - 90° and 30° - 60° - 90° Triangles

Topic 2 Trigonometric Ratios in the Coordinate Plane

Topic 3 Sine and Cosine Laws



Appendix B
Graphing Material and Table

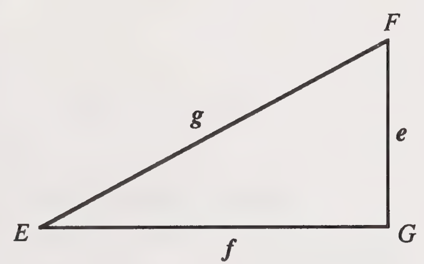
Graph Paper

Table of Trigonometric Functions

Appendix A
Solutions

Review

1.



2.
$$g^2 = e^2 + f^2$$
$$g^2 = (10\text{ cm})^2 + (7\text{ cm})^2$$
$$g^2 = 100\text{ cm}^2 + 49\text{ cm}^2$$
$$g^2 = 149\text{ cm}^2$$
$$g = \sqrt{149\text{ cm}^2}$$
$$g \doteq 12.2\text{ cm}$$

3.
$$g^2 = e^2 + f^2$$
$$f^2 = g^2 - e^2$$
$$f^2 = (10\text{ cm})^2 - (6\text{ cm})^2$$
$$f^2 = 100\text{ cm}^2 - 36\text{ cm}^2$$
$$f^2 = 64\text{ cm}^2$$
$$f = \sqrt{64\text{ cm}^2}$$
$$f = 8\text{ cm}$$

4.
$$g^2 = e^2 + f^2$$
$$e^2 = g^2 - f^2$$
$$e^2 = (27\text{ cm})^2 - (15\text{ cm})^2$$
$$e^2 = 729\text{ cm}^2 - 225\text{ cm}^2$$
$$e^2 = 504\text{ cm}^2$$
$$e = \sqrt{504\text{ cm}^2}$$
$$e \doteq 22.4\text{ cm}$$

5. a. $\sin A = \frac{8}{10}$
$= \frac{4}{5}$

b. $\sin B = \frac{6}{10}$
$= \frac{3}{5}$

c. $\csc A = \frac{10}{8}$
$= \frac{5}{4}$

d. $\csc B = \frac{10}{6}$
$= \frac{5}{3}$

e. $\cos A = \frac{6}{10}$

$= \frac{3}{5}$

f. $\cos B = \frac{8}{10}$

$= \frac{4}{5}$

g. $\sec A = \frac{10}{6}$

$= \frac{5}{3}$

h. $\sec B = \frac{10}{8}$

$= \frac{5}{4}$

i. $\tan A = \frac{8}{6}$

$= \frac{4}{3}$

j. $\tan B = \frac{6}{8}$

$= \frac{3}{4}$

k. $\cot A = \frac{6}{8}$

$= \frac{3}{4}$

l. $\cot B = \frac{8}{6}$

$= \frac{4}{3}$

6. Any one of the following methods can be used to find the measure of $\angle A$.

• $\sin A = \frac{4}{5}$

Enter	Display
4	4
÷	4
5	5
=	0.8
Inv	0.8
sin	53.13010236

$\angle A \doteq 53°$

• $\cos A = \frac{3}{5}$

Enter	Display
3	3
÷	3
5	5
=	0.6
Inv	0.6
cos	53.13010236

$\angle A \doteq 53°$

• $\tan A = \frac{4}{3}$

Enter	Display
4	4
÷	4
3	3
=	1.33333333
Inv	1.33333333
tan	53.13010236

$\angle A \doteq 53°$

- $\csc A = \frac{5}{4}$

Enter	Display
5	5
÷	5
4	4
=	1.25
1/x	0.8
Inv	0.8
sin	53.13010236

$\angle A \doteq 53°$

- $\sec A = \frac{5}{3}$

Enter	Display
5	5
÷	5
3	3
=	1.66666667
1/x	0.6
Inv	0.6
cos	53.13010236

$\angle A \doteq 53°$

- $\cot A = \frac{3}{4}$

Enter	Display
3	3
÷	3
4	4
=	0.75
1/x	1.33333333
Inv	1.33333333
tan	53.13010236

$\angle A \doteq 53°$

7. Any one of the six trigonometric ratios may be used. Only the method using the sine ratio will be shown.

$\sin B = \frac{3}{5}$

Enter	Display
3	3
÷	3
5	5
=	0.6
Inv	0.6
sin	36.86989765

$\angle B \doteq 37°$

8. The mathematical concepts dealing with trigonometry that you have studied so far are based on the third angle of the triangle being a right angle.

9. a. $\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{2\sqrt{3}}{3}$

b. $\frac{10}{\sqrt{5}} = \frac{10}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$

$= \frac{10\sqrt{5}}{5}$

$= 2\sqrt{5}$

c. $\frac{6}{5\sqrt{3}} = \frac{6}{5\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{6\sqrt{3}}{5\times 3}$

$= \frac{2\sqrt{3}}{5}$

d. $\frac{4}{3\sqrt{2}} = \frac{4}{3\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{4\sqrt{2}}{3\times 2}$

$= \frac{2\sqrt{2}}{3}$

Exploring Topic 1

Activity 1

Calculate the exact relative measures of the sides of 45° - 45° - 90° triangles.

1. The measure of the other side must also be 30 cm since both sides of an isosceles triangle are equal in length. The measure of the hypotenuse is $30\sqrt{2}$ cm since the length of the hypotenuse is $\sqrt{2}$ times the length of a side. (You could also solve this using the Pythagorean theorem.)

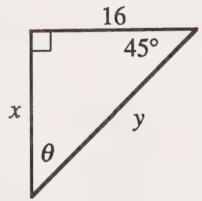
2. The lengths of the other sides will be as follows:

$$\frac{7\text{ m}}{\sqrt{2}} = \frac{7\text{ m}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{7\sqrt{2}\text{ m}}{2}$$

This isosceles triangle must be a 45° - 45° - 90° triangle since one angle is given as 90°. The remaining 90° must be equally divided since the triangle is isosceles.

3. a.

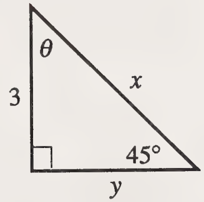


$x = 16$

$y = 16\sqrt{2}$

$\theta = 45°$

b.

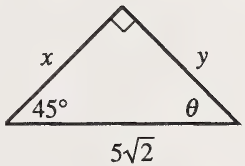


$x = 3\sqrt{2}$

$y = 3$

$\theta = 45°$

c.

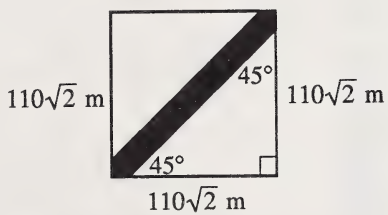


$x = 5$

$y = 5$

$\theta = 45°$

4.



The length of the sidewalk is $\left(110\sqrt{2}\text{ m}\right)\sqrt{2} = (110\text{ m})(2)$

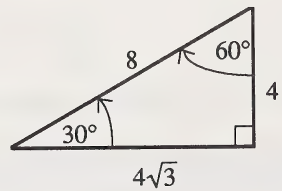
$= 220\text{ m}.$

Activity 2

Calculate the exact relative measures of the sides of 30° - 60° - 90° triangles.

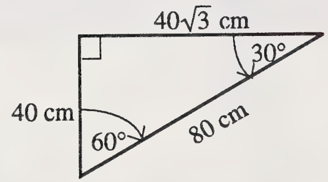
1. The measure of the shortest side is $\frac{4\sqrt{3}}{\sqrt{3}} = 4$.

   The measure of the hypotenuse is $2(4) = 8$.



2. The measure of the shortest side is $\frac{80\text{ cm}}{2} = 40\text{ cm}$.

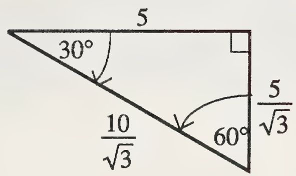
   The measure of the other side is $40\sqrt{3}$ cm.



3. The hypotenuse is $\left(\frac{5}{\sqrt{3}}\right)(2) = \frac{10}{\sqrt{3}}$

$$= \frac{10\sqrt{3}}{3}.$$

The other side measures $\left(\frac{5}{\sqrt{3}}\right)\left(\sqrt{3}\right) = 5.$

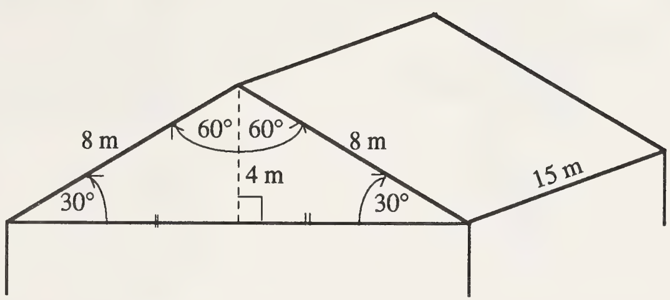


4. The length of each side of the roof is 8 m.

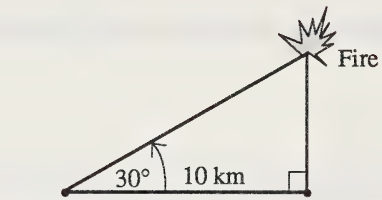
$2(4\text{ m}) = 8\text{ m}$

The total area of the entire roof is therefore

$2(8\text{ m})(15\text{ m}) = 240\text{ m}^2.$



5.



The distance to the fire from Fire Tower 2 is $\frac{10}{\sqrt{3}}$ km.

The distance to the fire from Fire Tower 1 is

$\left(\frac{10\text{ km}}{\sqrt{3}}\right)2 = \frac{20}{\sqrt{3}}$ km.

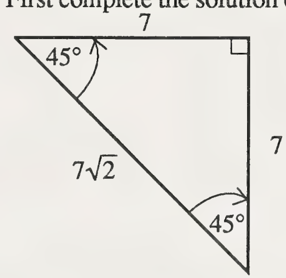
If you rationalize the denominators, you get

$\frac{10\sqrt{3}}{3}$ km and $\frac{20\sqrt{3}}{3}$ km, respectively.

Activity 3

Calculate the exact values of the trigonometric ratios for 45° - 45° - 90° triangles.

1. First complete the solution of the sides of the triangle.



$$\sin 45° = \frac{7}{7\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}}$$
$$= \frac{\sqrt{2}}{2}$$

2. $\csc 45° = \frac{7\sqrt{2}}{7}$
$= \sqrt{2}$

3. $\cos 45° = \frac{7}{7\sqrt{2}}$
$= \frac{1}{\sqrt{2}}$
$= \frac{\sqrt{2}}{2}$

4. $\sec 45° = \frac{7\sqrt{2}}{7}$
$= \sqrt{2}$

5. $\tan 45° = \frac{7}{7}$
$= 1$

6. $\cot 45° = \frac{7}{7}$
$= 1$

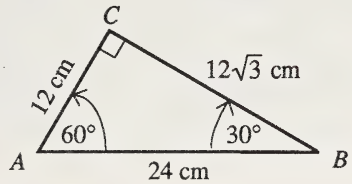
Activity 4

Calculate the exact values of the trigonometric ratios for 30° - 60° - 90° triangles.

1. To verify the exact trigonometric functions with your calculator, enter the value as calculated in this activity, and then use the calculator function key to compare the numbers.

Trigonometric Function	Exact Value	Calculated Decimal Approximation	Calculator Function
sin 30°	$\frac{1}{2}$	0.5	0.5
cos 30°	$\frac{\sqrt{3}}{2}$	0.866025403	0.866025403
tan 30°	$\frac{\sqrt{3}}{3}$	0.577350269	0.577350269
csc 30°	2	2	2
sec 30°	$\frac{2\sqrt{3}}{3}$	1.154700538	1.154700538
cot 30°	$\sqrt{3}$	1.732050808	1.732050808
sin 60°	$\frac{\sqrt{3}}{2}$	0.866025403	0.866025403
cos 60°	$\frac{1}{2}$	0.5	0.5
tan 60°	$\sqrt{3}$	1.732050808	1.732050808
csc 60°	$\frac{2\sqrt{3}}{3}$	1.154700538	1.154700538
sec 60°	2	2	2
cot 60°	$\frac{\sqrt{3}}{3}$	0.577350269	0.577350269

2. The calculated sides of this triangle are shown in the diagram.



$$\sin 30° = \frac{12 \text{ cm}}{24 \text{ cm}} = \frac{1}{2}$$

$$\cos 30° = \frac{12\sqrt{3} \text{ cm}}{24 \text{ cm}} = \frac{\sqrt{3}}{2}$$

$$\tan 30° = \frac{12 \text{ cm}}{12\sqrt{3} \text{ cm}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\csc 30° = \frac{24 \text{ cm}}{12 \text{ cm}} = 2$$

$$\sin 60° = \frac{12\sqrt{3} \text{ cm}}{24 \text{ cm}} = \frac{\sqrt{3}}{2}$$

$$\cos 60° = \frac{12 \text{ cm}}{24 \text{ cm}} = \frac{1}{2}$$

$$\tan 60° = \frac{12\sqrt{3} \text{ cm}}{12 \text{ cm}} = \sqrt{3}$$

$$\csc 60° = \frac{24 \text{ cm}}{12\sqrt{3} \text{ cm}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\sec 30° = \frac{24 \text{ cm}}{12\sqrt{3} \text{ cm}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\cot 30° = \frac{12\sqrt{3} \text{ cm}}{12 \text{ cm}} = \sqrt{3}$$

$$\sec 60° = \frac{24 \text{ cm}}{12 \text{ cm}} = 2$$

$$\cot 60° = \frac{12 \text{ cm}}{12\sqrt{3} \text{ cm}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

3. a. $\cos 60° = \underline{\mathbf{\frac{1}{2}}}$

b. $\sin 30° = \underline{\mathbf{\frac{1}{2}}}$

c. $\tan 60° = \underline{\sqrt{3}}$

d. $\tan \underline{\mathbf{30}}° = \frac{\sqrt{3}}{3}$

e. $\csc \underline{\mathbf{60}}° = \frac{2\sqrt{3}}{3}$

f. $\sec \underline{\mathbf{60}}° = 2$

g. $\cos 30° = \underline{\mathbf{\frac{\sqrt{3}}{2}}}$

h. $\sin \underline{\mathbf{60}}° = \frac{\sqrt{3}}{2}$

i. $\cot \underline{\mathbf{60}}° = \frac{\sqrt{3}}{3}$

j. $\cot 30° = \underline{\sqrt{3}}$

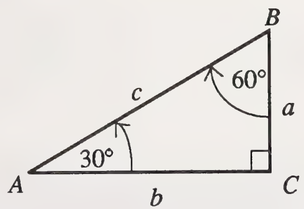
k. sec 30° = $\frac{2\sqrt{3}}{3}$ l. csc 30° = 2

If you had problems with the answers to e and i, you must remember how to rationalize denominators.

e. $\csc 60° = \frac{2}{\sqrt{3}}$
$= \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$
$= \frac{2\sqrt{3}}{3}$

i. $\cot 60° = \frac{1}{\sqrt{3}}$
$= \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$
$= \frac{\sqrt{3}}{3}$

Extra Help



sin A = $\frac{a}{c}$ sin B = $\frac{b}{c}$

csc A = $\frac{c}{a}$ csc B = $\frac{c}{b}$

cos A = $\frac{b}{c}$ cos B = $\frac{a}{c}$

sec A = $\frac{c}{b}$ sec B = $\frac{c}{a}$

tan A = $\frac{a}{b}$ tan B = $\frac{b}{a}$

cot A = $\frac{b}{a}$ cot B = $\frac{a}{b}$

Extensions

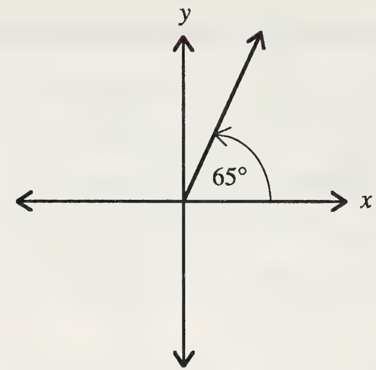
1. The ancient Egyptians discovered the magic 3-4-5 triangle.
2. The Pythagorean theorem was probably formulated by the Pythagorean school and credited to Pythagoras.
3. Answers will vary. Pythagoreanism may be defined as a philosophical school and a religious brotherhood.

Exploring Topic 2

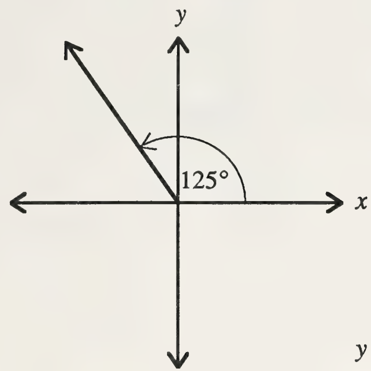
Activity 1

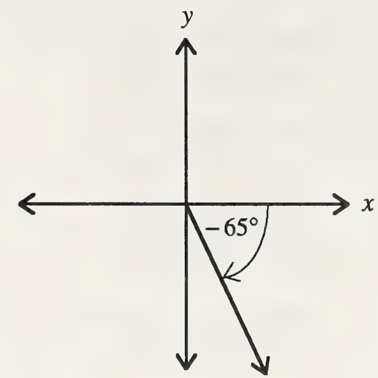
Recognize and sketch angles with positive and negative measures in standard position on a coordinate plane.

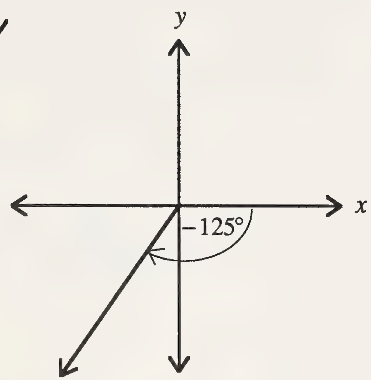
1. a. The terminal arm for 125° is in Quadrant II.
   The terminal arm for – 125° is in Quadrant III.
   The terminal arm for 65° is in Quadrant I.
   The terminal arm for – 65° is in Quadrant IV.
   The terminal arm for 400° is in Quadrant I.
   The terminal arm for – 400° is in Quadrant IV.
   The terminal arm for 275° is in Quadrant IV.
   The terminal arm for – 275° is in Quadrant I.

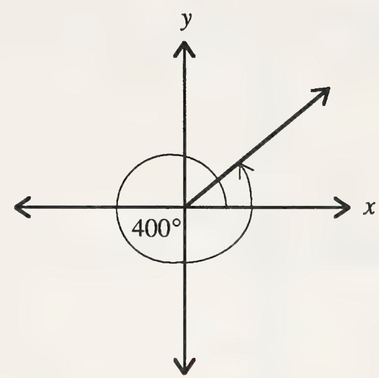


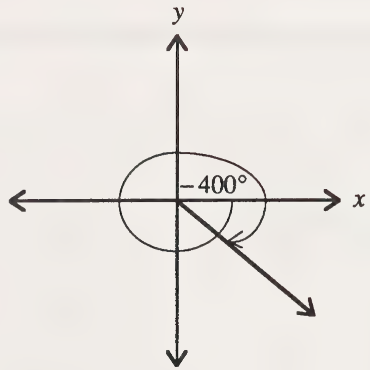
b.

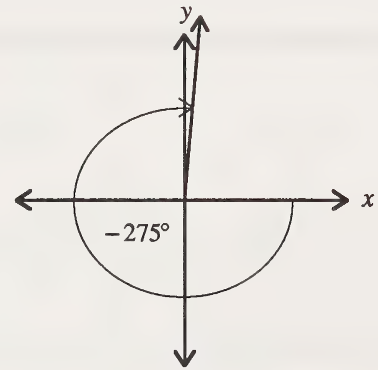


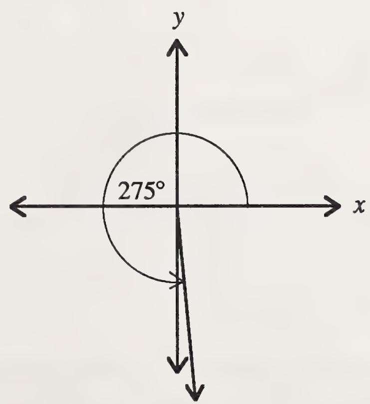












c. $125° + 360° = 485°$
$125° - 360° = -235°$

$-125° + 360° = 235°$
$-125° - 360° = -485°$

$65° + 360° = 425°$
$65° - 360° = -295°$

$-65° + 360° = 295°$
$-65° - 360° = -425°$

$400° - 360° = 40°$
$40° - 360° = -320°$

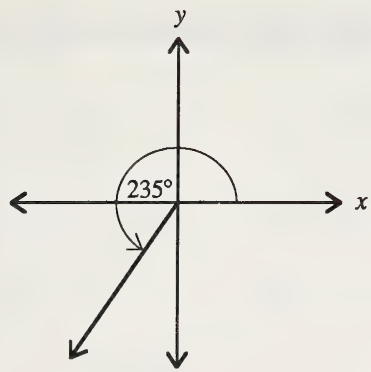
$-400° + 360° = -40°$
$-40° + 360° = 320°$

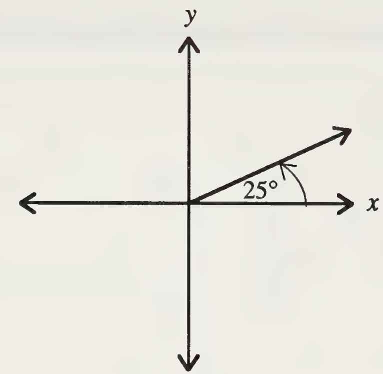
$275° + 360° = 635°$
$275° - 360° = -85°$

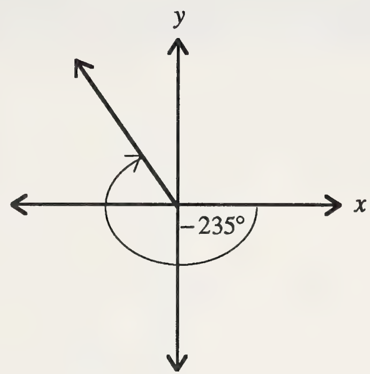
$-275° + 360° = 85°$
$-275° - 360° = -635°$

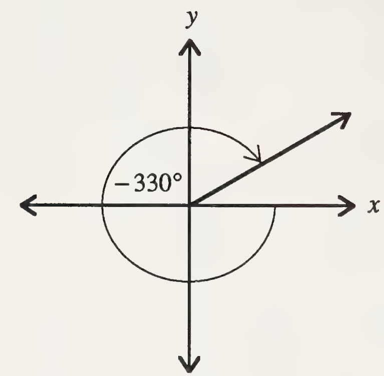
2. a. The terminal arm for 235° is in Quadrant III.
The terminal arm for – 235° is in Quadrant II.
The terminal arm for 25° is in Quadrant I.
The terminal arm for – 330° is in Quadrant I.
The terminal arm for 660° is in Quadrant IV.
The terminal arm for – 395° is in Quadrant IV.
The terminal arm for 110° is in Quadrant II.
The terminal arm for – 120° is in Quadrant III.

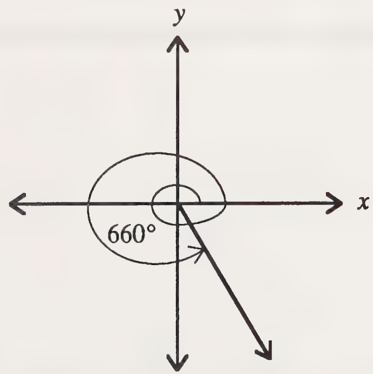
b.

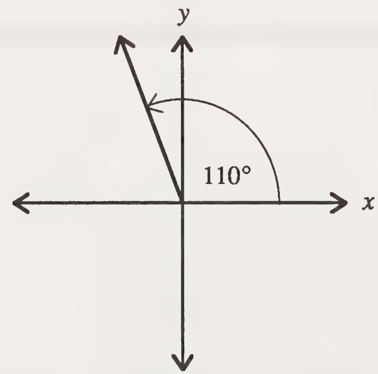


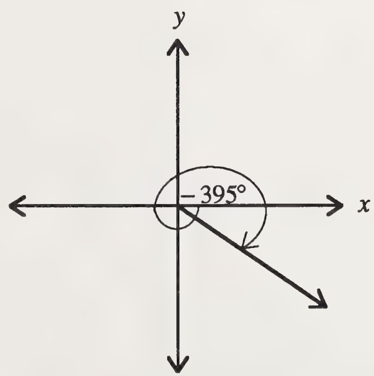


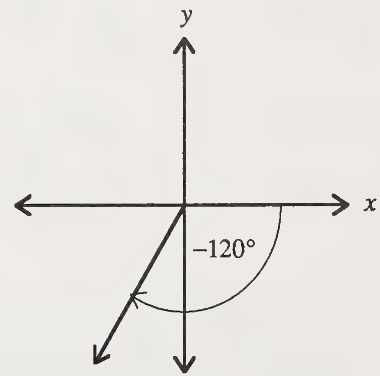





y
x
660°


y
x
110°


y
x
−395°


y
x
−120°

c. $235°+360°=595°$
$235°-360°=-125°$

$-235°+360°=125°$
$-235°-360°=-595°$

$25°+360°=385°$
$25°-360°=-335°$

$-330°+360°=30°$
$-330°-360°=-690°$

$660°-360°=300°$
$300°-360°=-60°$

$-395°+360°=-35°$
$-35°+360°=325°$

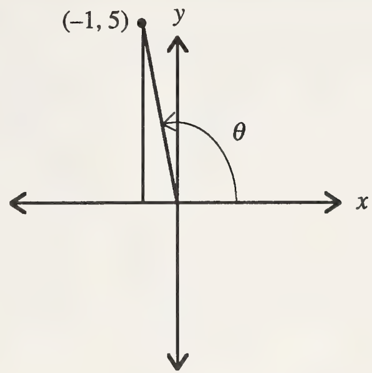
$110°+360°=470°$
$110°-360°=-250°$

$-120°+360°=240°$
$-120°-360°=-480°$

Activity 2

Express the trigonometric ratios and calculate the reference angle for an angle drawn in standard position on a coordinate plane.

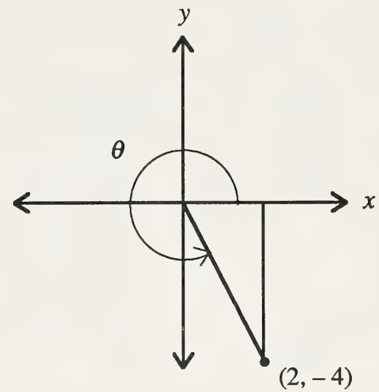
1. a.



$$r=\sqrt{(-1)^2+(5)^2}$$
$$=\sqrt{1+25}$$
$$=\sqrt{26}$$

$$\sin\theta=\frac{5}{\sqrt{26}}=\frac{5\sqrt{26}}{26}\qquad \cos\theta=\frac{-1}{\sqrt{26}}=\frac{-\sqrt{26}}{26}\qquad \tan\theta=-5$$

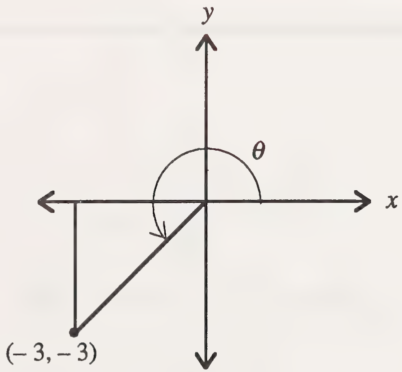
b.



$$r=\sqrt{(2)^2+(-4)^2}$$
$$=\sqrt{4+16}$$
$$=\sqrt{20}$$
$$=2\sqrt{5}$$

$$\sin\theta=\frac{-4}{2\sqrt{5}}=\frac{-2}{\sqrt{5}}=\frac{-2\sqrt{5}}{5}\qquad \cos\theta=\frac{2}{2\sqrt{5}}=\frac{1}{\sqrt{5}}=\frac{\sqrt{5}}{5}\qquad \tan\theta=\frac{-4}{2}=-2$$

c.



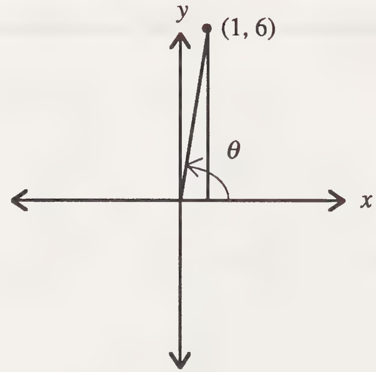
$$\begin{aligned} r &= \sqrt{(-3)^2 + (-3)^2} \\ &= \sqrt{9+9} \\ &= \sqrt{18} \\ &= 3\sqrt{2} \end{aligned}$$

$$\sin\theta = \frac{-3}{3\sqrt{2}} = \frac{-1}{\sqrt{2}} = \frac{-\sqrt{2}}{2}$$

$$\cos\theta = \frac{-3}{3\sqrt{2}} = \frac{-1}{\sqrt{2}} = \frac{-\sqrt{2}}{2}$$

$$\tan\theta = \frac{-3}{-3} = 1$$

d.



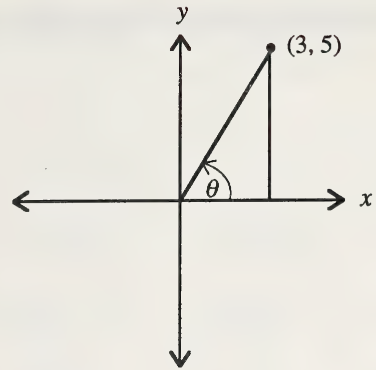
$$\begin{aligned} r &= \sqrt{1^2 + 6^2} \\ &= \sqrt{37} \end{aligned}$$

$$\sin\theta = \frac{6}{\sqrt{37}} = \frac{6\sqrt{37}}{37}$$

$$\cos\theta = \frac{1}{\sqrt{37}} = \frac{\sqrt{37}}{37}$$

$$\tan\theta = \frac{6}{1} = 6$$

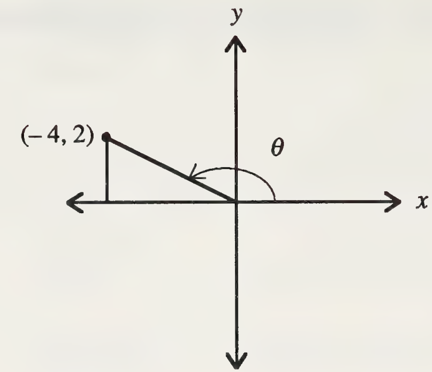
2. a.



$$r = \sqrt{3^2 + 5^2} = \sqrt{9+25} = \sqrt{34}$$

$$\sin\theta = \frac{5}{\sqrt{34}} = \frac{5\sqrt{34}}{34} \qquad \cos\theta = \frac{3}{\sqrt{34}} = \frac{3\sqrt{34}}{34} \qquad \tan\theta = \frac{5}{3}$$

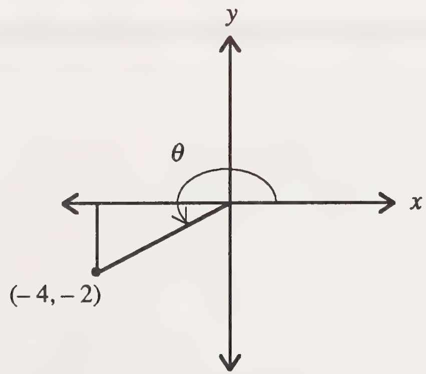
b.



$$r = \sqrt{(-4)^2 + 2^2} = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$$

$$\sin\theta = \frac{2}{2\sqrt{5}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5} \qquad \cos\theta = \frac{-4}{2\sqrt{5}} = \frac{-2}{\sqrt{5}} = \frac{-2\sqrt{5}}{5} \qquad \tan\theta = \frac{2}{-4} = -\frac{1}{2}$$

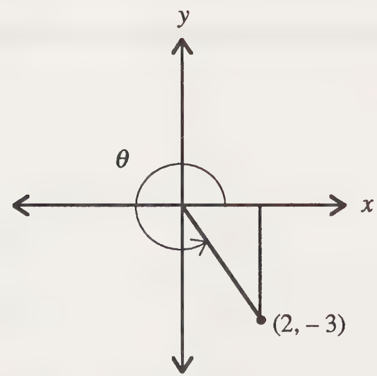
c.



$$r=\sqrt{(-4)^2+(-2)^2}$$
$$=\sqrt{16+4}$$
$$=\sqrt{20}$$
$$=2\sqrt{5}$$

$$\sin\ \theta=\frac{-2}{2\sqrt{5}} \qquad \cos\ \theta=\frac{-4}{2\sqrt{5}} \qquad \tan\ \theta=\frac{-2}{-4}$$
$$=\frac{-1}{\sqrt{5}} \qquad =\frac{-2}{\sqrt{5}} \qquad =\frac{1}{2}$$
$$=\frac{-\sqrt{5}}{5} \qquad =\frac{-2\sqrt{5}}{5}$$

d.



$$r=\sqrt{(2)^2+(-3)^2}$$
$$=\sqrt{4+9}$$
$$=\sqrt{13}$$

$$\sin\ \theta=\frac{-3}{\sqrt{13}} \qquad \cos\ \theta=\frac{2}{\sqrt{13}} \qquad \tan\ \theta=\frac{-3}{2}$$
$$=\frac{-3\sqrt{13}}{13} \qquad =\frac{2\sqrt{13}}{13}$$

3. a. i. In Quadrant III, $\cos\theta$ is negative.
   ii. In Quadrant IV, $\tan\theta$ is negative.
   iii. In Quadrant II, $\sin\theta$ is positive.

b. i. In Quadrant I, $\cos\theta$ and $\tan\theta$ are positive.
   ii. In Quadrant III, $\cos\theta$ and $\sin\theta$ are negative.
   iii. In Quadrant II, $\sin\theta$ is positive and $\tan\theta$ is negative.
   iv. In Quadrant III, $\cos\theta$ is negative and $\tan\theta$ is positive.

c. i. Since the terminal arm is in Quadrant III, $\cos 212°$ is negative.
   ii. Since the terminal arm is in Quadrant III, $\tan 181°$ is positive.
   iii. Since the terminal arm is in Quadrant III, $\sin 236°$ is negative.
   iv. Since the terminal arm is in Quadrant IV, $\cos 675°$ is positive.
   v. Since the terminal arm is in Quadrant IV, $\tan 317°$ is negative.
   vi. Since the terminal arm is in Quadrant IV, $\sin -62°$ is negative.
   vii. Since the terminal arm is in Quadrant II, $\cos 174°$ is negative.
   viii. Since the terminal arm is in Quadrant I, $\tan -295°$ is positive.
   ix. Since the terminal arm is in Quadrant I, $\sin 39°$ is positive.
   x. Since the terminal arm is in Quadrant II, $\sin -543°$ is positive.

4. a. i. In Quadrant III, $\sin\theta$ is negative.
   ii. In Quadrant IV, $\cos\theta$ is positive.
   iii. In Quadrant II, $\tan\theta$ is negative.

b. i. In Quadrant II, $\cos\theta$ and $\tan\theta$ are negative.
   ii. In Quadrant IV, $\sin\theta$ and $\tan\theta$ are negative.
   iii. In Quadrant III, $\sin\theta$ is negative and $\tan\theta$ is positive.
   iv. In Quadrant IV, $\cos\theta$ is positive and $\sin\theta$ is negative.

c. i. Since the terminal arm is in Quadrant I, $\cos 43°$ is positive.
   ii. Since the terminal arm is in Quadrant IV, $\tan 312°$ is negative.
   iii. Since the terminal arm is in Quadrant II, $\sin 499°$ is positive.
   iv. Since the terminal arm is in Quadrant II, $\cos 155°$ is negative.
   v. Since the terminal arm is in Quadrant II, $\tan -260°$ is negative.
   vi. Since the terminal arm is in Quadrant IV, $\sin -11°$ is negative.
   vii. Since the terminal arm is in Quadrant III, $\cos -127°$ is negative.
   viii. Since the terminal arm is in Quadrant II, $\tan 163°$ is negative.
   ix. Since the terminal arm is in Quadrant III, $\sin 260°$ is negative.
   x. Since the terminal arm is in Quadrant IV, $\sin 359°$ is negative.

5. a. The reference angle for 125° is 55°.

$|\sin 55°| \doteq 0.819\,152$

$|\sin 125°| \doteq 0.819\,152$

b. The reference angle for 65° is 65°.

$|\sin 65°| = |\sin 65°| \doteq 0.906\,308$

c. The reference angle for 350° is 10°.

$|\sin 350°| \doteq 0.173\,648$

$|\sin 10°| \doteq 0.173\,648$

d. Reference angle for 230° is 50°.

$|\sin 230°| \doteq 0.766\,044$

$|\sin 50°| \doteq 0.766\,044$

6. a.
$$\begin{aligned}\sin 98° &= \sin(180° - 98°)\\ &= \sin 82°\end{aligned}$$

$$\begin{aligned}\cos 98° &= -\cos(180° - 98°)\\ &= -\cos 82°\end{aligned}$$

$$\begin{aligned}\tan 98° &= -\tan(180° - 98°)\\ &= -\tan 82°\end{aligned}$$

b.
$$\begin{aligned}\sin 263° &= -\sin(263° - 180°)\\ &= -\sin 83°\end{aligned}$$

$$\begin{aligned}\cos 263° &= -\cos(263° - 180°)\\ &= -\cos 83°\end{aligned}$$

$$\begin{aligned}\tan 263° &= \tan(263° - 180°)\\ &= \tan 83°\end{aligned}$$

c.
$$\begin{aligned}\sin 345° &= -\sin(360° - 345°)\\ &= -\sin 15°\end{aligned}$$

$$\begin{aligned}\cos 345° &= \cos(360° - 345°)\\ &= \cos 15°\end{aligned}$$

$$\begin{aligned}\tan 345° &= -\tan(360° - 345°)\\ &= -\tan 15°\end{aligned}$$

d.
$$\begin{aligned}\sin -77° &= \sin(-77° + 360°)\\ &= \sin 283°\\ &= -\sin(360° - 283°)\\ &= -\sin 77°\end{aligned}$$

$$\begin{aligned}\cos -77^\circ &= \cos(-77^\circ + 360^\circ)\\ &= \cos 283^\circ\\ &= \cos(360^\circ - 283^\circ)\\ &= \cos 77^\circ\end{aligned}$$

$$\begin{aligned}\tan -77^\circ &= \tan(-77^\circ + 360^\circ)\\ &= \tan 283^\circ\\ &= -\tan(360^\circ - 283^\circ)\\ &= -\tan 77^\circ\end{aligned}$$

e. $$\begin{aligned}\sin -488^\circ &= \sin(-488^\circ + 720^\circ)\\ &= \sin 232^\circ\\ &= -\sin(232^\circ - 180^\circ)\\ &= -\sin 52^\circ\end{aligned}$$

$$\begin{aligned}\cos -488^\circ &= \cos(-488^\circ + 720^\circ)\\ &= \cos 232^\circ\\ &= -\cos(232^\circ - 180^\circ)\\ &= -\cos 52^\circ\end{aligned}$$

$$\begin{aligned}\tan -488^\circ &= \tan(-488^\circ + 720^\circ)\\ &= \tan 232^\circ\\ &= \tan(232^\circ - 180^\circ)\\ &= \tan 52^\circ\end{aligned}$$

f. $$\begin{aligned}\sin 569^\circ &= \sin(569^\circ - 360^\circ)\\ &= \sin 209^\circ\\ &= -\sin(209^\circ - 180^\circ)\\ &= -\sin 29^\circ\end{aligned}$$

$$\begin{aligned}\cos 569^\circ &= \cos(569^\circ - 360^\circ)\\ &= \cos 209^\circ\\ &= -\cos(209^\circ - 180^\circ)\\ &= -\cos 29^\circ\end{aligned}$$

$$\begin{aligned}\tan 569^\circ &= \tan(569^\circ - 360^\circ)\\ &= \tan 209^\circ\\ &= \tan(209^\circ - 180^\circ)\\ &= \tan 29^\circ\end{aligned}$$

g. $\sin 49^\circ$
$\cos 49^\circ$
$\tan 49^\circ$

7. a. $\sin 63^\circ$
$\cos 63^\circ$
$\tan 63^\circ$

b. $$\begin{aligned}\sin 149^\circ &= \sin(180^\circ - 149^\circ)\\ &= \sin 31^\circ\end{aligned}$$

$$\begin{aligned}\cos 149^\circ &= -\cos(180^\circ - 149^\circ)\\ &= -\cos 31^\circ\end{aligned}$$

$$\begin{aligned}\tan 149^\circ &= -\tan(180^\circ - 149^\circ)\\ &= -\tan 31^\circ\end{aligned}$$

c. $$\begin{aligned}\sin 186^\circ &= \sin(186^\circ - 180^\circ)\\ &= -\sin 6^\circ\end{aligned}$$

$$\begin{aligned}\cos 186^\circ &= -\cos(186^\circ - 180^\circ)\\ &= -\cos 6^\circ\end{aligned}$$

$$\begin{aligned}\tan 186^\circ &= \tan(186^\circ - 180^\circ)\\ &= \tan 6^\circ\end{aligned}$$

d. $$\begin{aligned}\sin 349^\circ &= -\sin(360^\circ - 349^\circ)\\ &= -\sin 11^\circ\end{aligned}$$

$$\begin{aligned}\cos 349^\circ &= \cos(360^\circ - 349^\circ)\\ &= \cos 11^\circ\end{aligned}$$

$$\begin{aligned}\tan 349^\circ &= -\tan(360^\circ - 349^\circ)\\ &= -\tan 11^\circ\end{aligned}$$

e. $$\begin{aligned}\sin -20^\circ &= \sin(-20^\circ + 360^\circ)\\ &= \sin 340^\circ\\ &= -\sin(360^\circ - 340^\circ)\\ &= -\sin 20^\circ\end{aligned}$$

$$\begin{aligned}\cos -20^\circ &= \cos(-20^\circ + 360^\circ)\\ &= \cos 340^\circ\\ &= \cos(360^\circ - 340^\circ)\\ &= \cos 20^\circ\end{aligned}$$

$$\begin{aligned}\tan -20^\circ &= \tan(-20^\circ + 360^\circ)\\ &= \tan 340^\circ\\ &= -\tan(360^\circ - 340^\circ)\\ &= -\tan 20^\circ\end{aligned}$$

f. $$\begin{aligned}\sin -592^\circ &= \sin(-592^\circ + 720^\circ)\\ &= \sin 128^\circ\\ &= \sin(180^\circ - 128^\circ)\\ &= \sin 52^\circ\end{aligned}$$

$$\begin{aligned}\cos -592^\circ &= \cos(-592^\circ + 720^\circ)\\ &= \cos 128^\circ\\ &= -\cos(180^\circ - 128^\circ)\\ &= -\cos 52^\circ\end{aligned}$$

$$\begin{aligned}\tan -592^\circ &= \tan(-592^\circ + 720^\circ)\\ &= \tan 128^\circ\\ &= -\tan(180^\circ - 128^\circ)\\ &= -\tan 52^\circ\end{aligned}$$

g. $$\begin{aligned}\sin -320^\circ &= \sin(-320^\circ + 360^\circ)\\ &= \sin 40^\circ\end{aligned}$$

$$\begin{aligned}\cos -320^\circ &= \cos(-320^\circ + 360^\circ)\\ &= \cos 40^\circ\end{aligned}$$

$$\begin{aligned}\tan -320^\circ &= \tan(-320^\circ + 360^\circ)\\ &= \tan 40^\circ\end{aligned}$$

Activity 3

Calculate the trigonometric ratios for any angle.

1. a. $$\begin{aligned}\sin 163^\circ &= \sin(180^\circ - 163^\circ)\\ &= \sin 17^\circ\\ &\doteq 0.2924\end{aligned}$$

$$\begin{aligned}\cos 163^\circ &= -\cos(180^\circ - 163^\circ)\\ &= -\cos 17^\circ\\ &\doteq -0.9563\end{aligned}$$

$$\begin{aligned}\tan 163^\circ &= -\tan(180^\circ - 163^\circ)\\ &= -\tan 17^\circ\\ &\doteq -0.3057\end{aligned}$$

b. $$\begin{aligned}\sin 92^\circ &= \sin(180^\circ - 92^\circ)\\ &= \sin 88^\circ\\ &\doteq 0.9994\end{aligned}$$

$$\begin{aligned}\cos 92^\circ &= -\cos(180^\circ - 92^\circ)\\ &= -\cos 88^\circ\\ &\doteq -0.0349\end{aligned}$$

$$\begin{aligned}\tan 92^\circ &= -\tan(180^\circ - 92^\circ)\\ &= -\tan 88^\circ\\ &\doteq -28.6363\end{aligned}$$

c. $\sin 270° = -\sin(360° - 270°)$
$= -\sin 90°$
$= -1$

$\cos 270° = \cos(360° - 270°)$
$= \cos 90°$
$= 0$

$\tan 270° = -\tan(360° - 270°)$
$= -\tan 90°$
undefined

d. $\sin -321° = \sin(-321° + 360°)$
$= \sin 39°$
$\doteq 0.6293$

$\cos -321° = \cos(-321° + 360°)$
$= \cos 39°$
$\doteq 0.7771$

$\tan -321° = \tan(-321° + 360°)$
$= \tan 39°$
$\doteq 0.8098$

e. $\sin 660° = \sin(660° - 360°)$
$= \sin 300°$
$= -\sin(360° - 300°)$
$= -\sin 60°$
$\doteq -0.8660$

$\cos 660° = \cos(660° - 360°)$
$= \cos 300°$
$= \cos(360° - 300°)$
$= \cos 60°$
$= 0.5000$

$\tan 660° = \tan(660° - 360°)$
$= \tan 300°$
$= -\tan(360° - 300°)$
$= -\tan 60°$
$\doteq -1.7321$

f. $\sin 23° \doteq 0.3907$
$\cos 23° \doteq 0.9205$
$\tan 23° \doteq 0.4245$

2\. a.

$$\sin 31° \doteq 0.5150$$
$$\cos 31° \doteq 0.8572$$
$$\tan 31° \doteq 0.6009$$

b.

$$\begin{aligned}\sin -31° &= \sin(-31° + 360°)\\ &= \sin 329°\\ &= -\sin(360° - 329°)\\ &= -\sin 31°\\ &\doteq -0.5150\end{aligned}$$

$$\begin{aligned}\cos -31° &= \cos(-31° + 360°)\\ &= \cos 329°\\ &= \cos(360° - 329°)\\ &= \cos 31°\\ &\doteq 0.8572\end{aligned}$$

$$\begin{aligned}\tan -31° &= \tan(-31° + 360°)\\ &= \tan 329°\\ &= -\tan(360° - 329°)\\ &= -\tan 31°\\ &\doteq -0.6009\end{aligned}$$

c.

$$\begin{aligned}\sin 180° &= \sin(180° - 180°)\\ &= \sin 0°\\ &= 0\end{aligned}$$

$$\begin{aligned}\cos 180° &= \cos(180° - 180°)\\ &= -\cos 0°\\ &= -1\end{aligned}$$

$$\begin{aligned}\tan 180° &= -\tan(180° - 180°)\\ &= -\tan 0°\\ &= 0\end{aligned}$$

d.

$$\begin{aligned}\sin 193° &= -\sin(193° - 180°)\\ &= -\sin 13°\\ &\doteq -0.2250\end{aligned}$$

$$\begin{aligned}\cos 193° &= -\cos(193° - 180°)\\ &= -\cos 13°\\ &\doteq -0.9744\end{aligned}$$

$$\begin{aligned}\tan 193° &= \tan(193° - 180°)\\ &= \tan 13°\\ &\doteq 0.2309\end{aligned}$$

e. $$\begin{aligned}\sin 491° &= \sin(491° - 360°)\\ &= \sin 131°\\ &= \sin(180° - 131°)\\ &= \sin 49°\\ &\doteq 0.7547\end{aligned}$$

$$\begin{aligned}\cos 491° &= \cos(491° - 360°)\\ &= \cos 131°\\ &= -\cos(180° - 131°)\\ &= -\cos 49°\\ &\doteq -0.6561\end{aligned}$$

$$\begin{aligned}\tan 491° &= \tan(491° - 360°)\\ &= \tan 131°\\ &= -\tan(180° - 131°)\\ &= -\tan 49°\\ &\doteq 1.1504\end{aligned}$$

f. $$\begin{aligned}\sin -375° &= \sin(-375° + 720°)\\ &= \sin 345°\\ &= -\sin(360° - 345°)\\ &= -\sin 15°\\ &\doteq -0.2588\end{aligned}$$

$$\begin{aligned}\cos -375° &= \cos(-375° + 720°)\\ &= \cos 345°\\ &= \cos(360° - 345°)\\ &= \cos 15°\\ &\doteq 0.9659\end{aligned}$$

$$\begin{aligned}\tan -375° &= \tan(-375° + 720°)\\ &= \tan 345°\\ &= -\tan(360° - 345°)\\ &= -\tan 15°\\ &\doteq -0.2679\end{aligned}$$

3. a.

Enter	Display
647	647
$\boxed{+/-}$	−647
$\boxed{\sin}$	0.956304756

$\sin -647° \doteq 0.956\,304\,756$

Enter	Display
647	647
$\boxed{+/-}$	−647
$\boxed{\cos}$	0.292371704

$\cos -647° \doteq 0.292\,371\,704$

Enter	Display
647	647
$\boxed{+/-}$	−647
$\boxed{\text{tan}}$	3.270852619

$\tan -647° \doteq 3.270\,852\,619$

b.

Enter	Display
317	317
$\boxed{\text{sin}}$	−0.681 998 36

$\sin 317° \doteq -0.681\,998\,36$

Enter	Display
317	317
$\boxed{\text{cos}}$	0.731353701

$\cos 317° \doteq 0.731\,353\,701$

Enter	Display
317	317
$\boxed{\text{tan}}$	−0.932515086

$\tan 317° \doteq -0.932\,515\,086$

c.

Enter	Display
21	21
$\boxed{\text{sin}}$	0.358367949

$\sin 21° \doteq 0.358\,367\,949$

Enter	Display
21	21
$\boxed{\text{cos}}$	0.933580426

$\cos 21° \doteq 0.933\,580\,426$

Enter	Display
21	21
$\boxed{\text{tan}}$	0.383864035

$\tan 21° \doteq 0.383\,864\,035$

d.

Enter	Display
180	180
$\boxed{\text{sin}}$	0

$\sin 180° = 0$

Enter	Display
180	180
$\boxed{\cos}$	−1

$\cos 180° = -1$

Enter	Display
180	180
$\boxed{\tan}$	0

$\tan 180° = 0$

e.

Enter	Display
217	217
$\boxed{\sin}$	−0.601815023

$\sin 217° \doteq -0.601\,815\,023$

Enter	Display
217	217
$\boxed{\cos}$	−0.79863551

$\cos 217° \doteq -0.798\,635\,51$

Enter	Display
217	217
$\boxed{\tan}$	0.75355405

$\tan 217° \doteq 0.753\,554\,05$

4. a.

Enter	Display
379	379
$\boxed{\sin}$	0.325568154

$\sin 379° \doteq 0.325\,568\,154$

Enter	Display
379	379
$\boxed{\cos}$	0.945518575

$\cos 379° \doteq 0.945\,518\,575$

Enter	Display
379	379
$\boxed{\tan}$	0.344327613

$\tan 379° \doteq 0.344\,327\,613$

b.

Enter	Display
95	95
$\boxed{+/-}$	−95
$\boxed{\text{sin}}$	−0.996194698

$\sin -95° \doteq -0.996\,194\,698$

Enter	Display
95	95
$\boxed{+/-}$	−95
$\boxed{\text{cos}}$	−0.087155742

$\cos -95° \doteq -0.087\,155\,742$

Enter	Display
95	95
$\boxed{+/-}$	−95
$\boxed{\text{tan}}$	11.4300523

$\tan -95° \doteq 11.430\,052\,3$

c.

Enter	Display
275	275
$\boxed{\text{sin}}$	−0.996194698

$\sin 275° \doteq -0.996\,194\,698$

Enter	Display
275	275
$\boxed{\text{cos}}$	0.087155742

$\cos 275° \doteq 0.087\,155\,742$

Enter	Display
275	275
$\boxed{\text{tan}}$	−11.4300523

$\tan 275° \doteq -11.430\,052\,3$

d.

Enter	Display
183	183
$\boxed{\text{sin}}$	−0.052335956

$\sin 183° \doteq -0.052\,335\,956$

Enter	Display
183	183
cos	−0.998629534

$\cos 183° \doteq -0.998\,629\,534$

Enter	Display
183	183
tan	0.052407779

$\tan 183° \doteq 0.052\,407\,779$

e.

Enter	Display
175	175
sin	0.087155742

$\sin 175° \doteq 0.087\,155\,742$

Enter	Display
175	175
cos	−0.996194698

$\cos 175° \doteq -0.996\,194\,698$

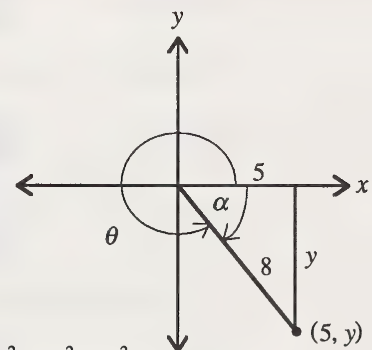
Enter	Display
175	175
tan	−0.087488663

$\tan 175° \doteq -0.087\,488\,663$

Activity 4

Determine any two values of x, y, r, and θ given the other two.

1. a.



$r^2 = x^2 + y^2$

$y^2 = r^2 - x^2$

$y^2 = 8^2 - 5^2$

$y^2 = 64 - 25$

$y^2 = 39$

$y \doteq -6.2$

(In Quadrant IV, y is negative.)

$\cos\alpha = \frac{5}{8}$

$\cos\alpha = 0.625$

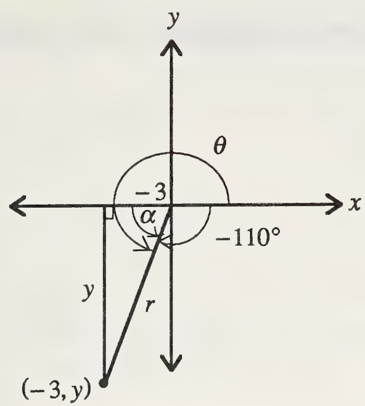
$\alpha \doteq 51°$

$\theta = 360° - \alpha$

$\theta = 360° - 51°$

$\theta = 309°$

b.



$\theta = 360° - 110°$

$\theta = 250°$

$\tan\ \theta = \frac{y}{-3}$

$y = -3 \times \tan\ \theta$

$y = -3 \times \tan\ 250°$

$y \doteq -3 \times 2.7475$

$y \doteq -8.2$

$\cos\ \theta = \frac{-3}{r}$

$r = \frac{-3}{\cos\ \theta}$

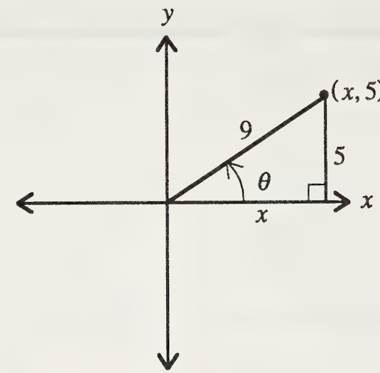
$r = \frac{-3}{\cos\ 250°}$

$r \doteq \frac{-3}{-0.3420}$

$r \doteq 8.8$

When you work with θ, the x- and y-values can be positive or negative; but for the angle α, x and y must be positive.

c.



$r^2 = x^2 + y^2$

$x^2 = r^2 - y^2$

$x^2 = 9^2 - 5^2$

$x^2 = 81 - 25$

$x^2 = 56$

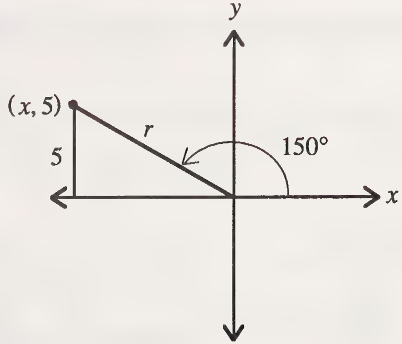
$x \doteq 7.5$

$\sin\ \theta = \frac{5}{9}$

$\sin\ \theta \doteq 0.5556$

$\theta \doteq 34°$

d.



$\sin 150° = \dfrac{5}{r}$

$r = \dfrac{5}{\sin 150°}$

$r = \dfrac{5}{0.5}$

$r = 10$

$\tan 150° = \dfrac{5}{x}$

$x = \dfrac{5}{\tan 150°}$

$x = \dfrac{5}{-0.5774}$

$x \doteq -8.7$

e. $x = -5$ and $y = -3$

When working with α, the x- and y-values must be positive.

$\tan \alpha = \dfrac{y}{x}$

$\tan \alpha = \dfrac{3}{5}$

$\tan \alpha = 0.6$

$\alpha \doteq 31°$

Since this is a third-quadrant angle, use the following formula:

$\theta = 180 + \alpha$

$\theta = 180 + 31°$

$\theta = 211°$

$r^2 = x^2 + y^2$

$r^2 = (-5)^2 + (-3)^2$

$r^2 = 25 + 9$

$r^2 = 34$

$r \doteq 5.8$

f. This is a second-quadrant angle; therefore, sine is positive and cosine is negative.

$\cos 144° = \dfrac{x}{r}$

$\cos 144° = \dfrac{x}{15}$

$x = 15 \times \cos 144°$

$x \doteq 15 \times -0.8090$

$x \doteq -12.1$

$\sin 144° = \dfrac{y}{r}$

$\sin 144° = \dfrac{y}{15}$

$y = 15 \times \sin 144°$

$y \doteq 15 \times 0.5879$

$y \doteq 8.8$

g. Since tangent is positive and sine is negative, this is a third-quadrant angle. Cosine is negative in Quadrant III.

$$\tan\ \theta = \frac{y}{x} \text{ and } \tan\ \theta = \frac{12}{5}$$
$$\tan\ \theta = \tan\ \theta$$
$$\frac{y}{x} = \frac{12}{5}$$

Let $y = -12$; then $x = -5$.

$$r^2 = x^2 + y^2$$
$$r^2 = (-5)^2 + (-12)^2$$
$$r^2 = 25 + 144$$
$$r^2 = 169$$
$$r = 13$$

One possible terminal point is $(-5, -12)$.

$$\tan\ \alpha = \frac{12}{5}$$
$$\tan\ \alpha = 2.4$$
$$\alpha \doteq 67°$$

Since this angle is in Quadrant III, use the following formula:

$$\theta = 180° + \alpha$$
$$\theta = 180° + 67°$$
$$\theta = 247°$$

h. $\sin\ \theta = \frac{-2}{3}$ and $\sin\ \theta = \frac{y}{r}$

$$\sin\ \theta = \sin\ \theta$$
$$\frac{-2}{3} = \frac{y}{r}$$

Let $y = -2$ and $r = 3$.

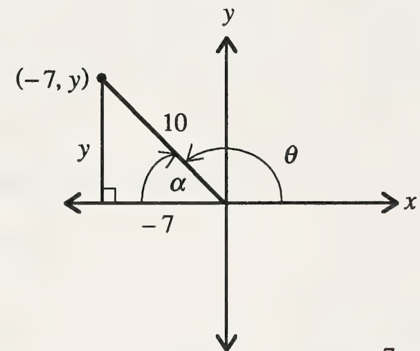
$$r^2 = x^2 + y^2$$
$$x^2 = r^2 - y^2$$
$$x^2 = 3^2 - (-2)^2$$
$$x^2 = 9 - 4$$
$$x^2 = 5$$
$$x \doteq 2.2$$

$$\sin\ \alpha = \frac{2}{3}$$
$$\sin\ \alpha \doteq 0.6667$$
$$\alpha \doteq 42°$$

$$\theta = 360° - \alpha$$
$$\theta \doteq 360° - 42°$$
$$\theta \doteq 318°$$

One possible terminal point is $(2.2, -2)$ which gives an r-value of 3.

2. a.

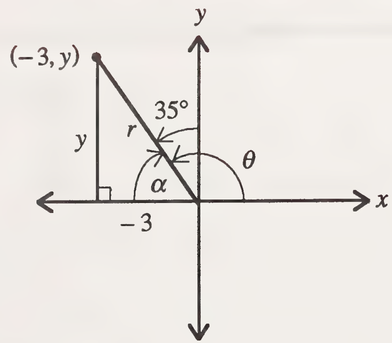


$$r^2 = x^2 + y^2$$
$$y^2 = r^2 - x^2$$
$$y^2 = 10^2 - (-7)^2$$
$$y^2 = 100 - 49$$
$$y^2 = 51$$
$$y \doteq 7.1$$

$$\cos\ \alpha = \frac{7}{10}$$
$$\cos\ \alpha = 0.7$$
$$\alpha \doteq 46°$$

$$\theta = 180° - \alpha$$
$$\theta = 180° - 46°$$
$$\theta = 134°$$

b.



$\theta = 90° + 35°$

$\quad = 125°$

$\cos 125° = \dfrac{-3}{r}$

$r = \dfrac{-3}{\cos 125°}$

$r \doteq \dfrac{-3}{-0.5736}$

$r \doteq 5.2$

(Cosine is negative in Quadrant II.)

$\tan 125° = \dfrac{y}{-3}$

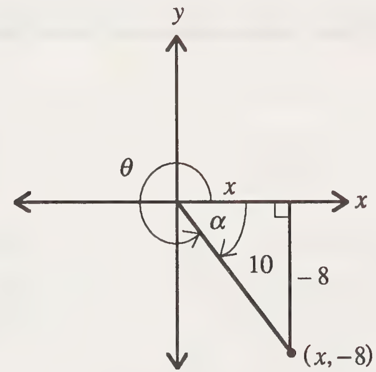
$y = -3 \times \tan 125°$

$y \doteq -3 \times -1.4281$

$y \doteq 4.3$

(Tangent is negative in Quadrant II.)

c.



$r^2 = x^2 + y^2$

$x^2 = r^2 - y^2$

$x^2 = 10^2 - (-8)^2$

$x^2 = 100 - 64$

$x^2 = 36$

$x = 6$

$\sin \alpha = \dfrac{8}{10}$

$\sin \alpha = 0.8$

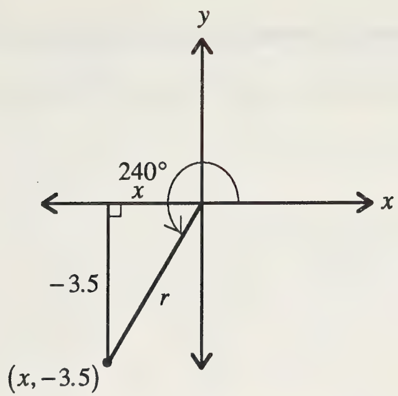
$\alpha \doteq 53°$

$\theta = 360° - \alpha$

$\theta = 360° - 53°$

$\theta = 307°$

d.



$$\sin 240° = \frac{-3.5}{r}$$
$$r = \frac{-3.5}{\sin 240°}$$
$$r \doteq \frac{-3.5}{-0.8660}$$
$$r \doteq 4.0$$

$$\tan 240° = \frac{-3.5}{x}$$
$$x = \frac{-3.5}{\tan 240°}$$
$$x \doteq \frac{-3.5}{1.7321}$$
$$x \doteq -2.0$$

e. $x = 4$ and $y = -7$

$$r^2 = x^2 + y^2$$
$$r^2 = 4^2 + (-7)^2$$
$$r^2 = 16 + 49$$
$$r^2 = 65$$
$$r \doteq 8.1$$

$$\tan \alpha = \frac{-7}{4}$$
$$\tan \alpha = -1.75$$
$$\alpha = -60°$$
(ignore the sign)

This angle is in Quadrant IV.

$$\theta = 360° - \alpha$$
$$\theta = 360° - 60°$$
$$\theta = 300°$$

f. This angle is in Quadrant II.

$$\theta = -220°$$

$$\sin -220° = \frac{y}{12}$$
$$y = 12 \times \sin -220°$$
$$y \doteq 12 \times 0.6428$$
$$y \doteq 7.7$$

$$\cos -220° = \frac{x}{12}$$
$$x = 12 \times \cos -220°$$
$$x \doteq 12 \times -0.7660$$
$$x \doteq -9.2$$

g. θ is in Quadrant II.

$\tan\ \theta = \frac{y}{x}$ and

$\tan\ \theta = \frac{-12}{5}$

$\tan\ \theta = \tan\ \theta$

$\frac{y}{x} = \frac{-12}{5}$

Let $x = -5$ and $y = 12$. (x must be negative to make cosine negative.)

One possible terminal point is $(-5,\ 12)$.

$$r^2 = x^2 + y^2$$
$$r^2 = (-5)^2 + 12^2$$
$$r^2 = 25 + 144$$
$$r^2 = 169$$
$$r = 13$$

$$\tan\ \alpha = \frac{12}{5}$$
$$\tan\ \alpha = 2.4$$
$$\alpha \doteq 67°$$

$$\theta = 180° - \alpha$$
$$\theta = 180° - 67°$$
$$\theta = 113°$$

h. $\cos\ \theta = \frac{x}{r}$ and $\cos\ \theta = \frac{-3}{4}$

$\cos\ \theta = \cos\ \theta$

$\frac{x}{r} = \frac{-3}{4}$

Let $x = -3$

and $r = 4$.

$$r^2 = x^2 + y^2$$
$$y^2 = r^2 - x^2$$
$$y^2 = 4^2 - (-3)^2$$
$$y^2 = 16 - 9$$
$$y^2 = 7$$
$$y \doteq 2.6$$

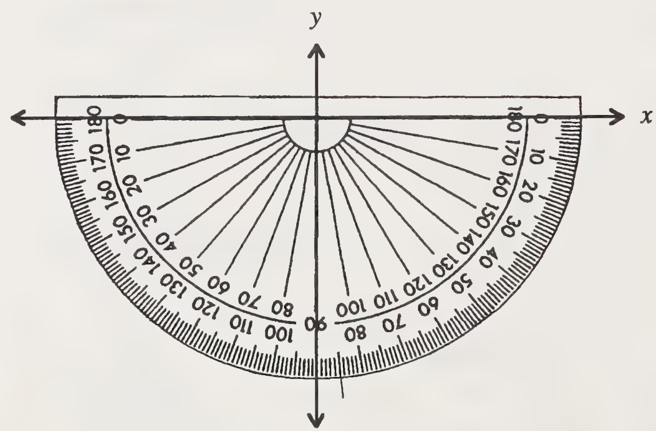
$$\cos\ \alpha = \frac{3}{4}$$
$$\cos\ \alpha = 0.75$$
$$\alpha \doteq 41°$$
$$\theta = 180° - \alpha$$
$$\theta = 180° - 41°$$
$$\theta = 139°$$

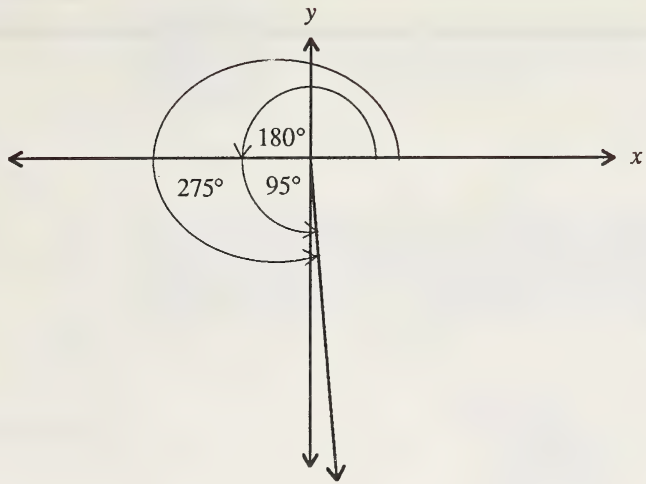
One possible terminal point is $(-3, 2.6)$ which gives an r-value of 4.

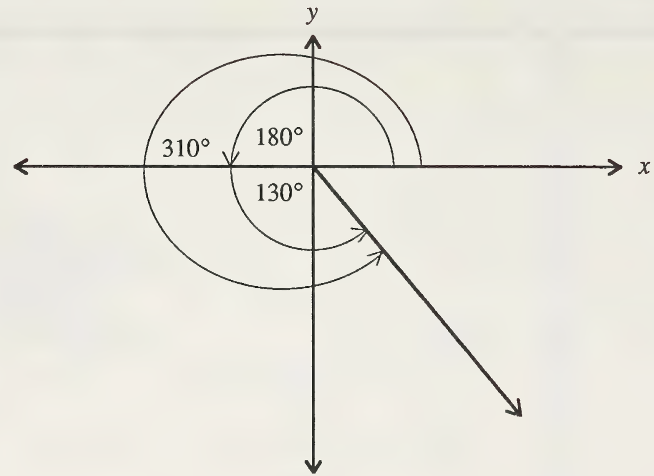
Extra Help

1. a. $345° - 180° = 165°$
   b. $195° - 180° = 15°$
   c. $270° - 180° = 90°$
   d. $290° - 180° = 110°$
   e. $210° - 180° = 30°$
   f. $355° - 180° = 175°$
   g. $175°$ (This angle is less than $180°$.)
   h. $360° - 180° = 180°$ ($360°$ is exactly the same as $0°$.)

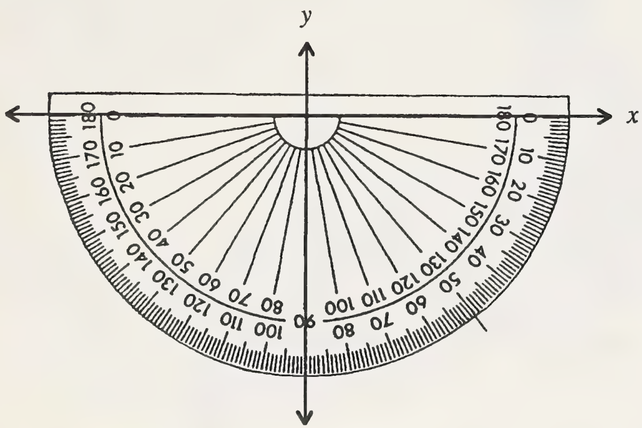
2. a. $275° - 180° = 95°$
   Measure $95°$ from the negative x-axis.



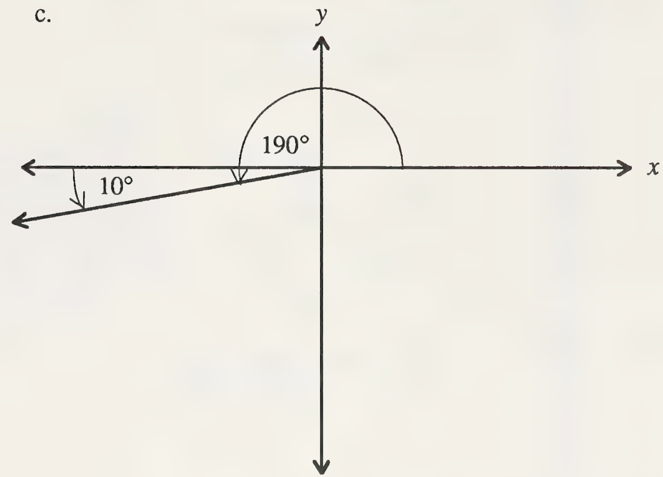




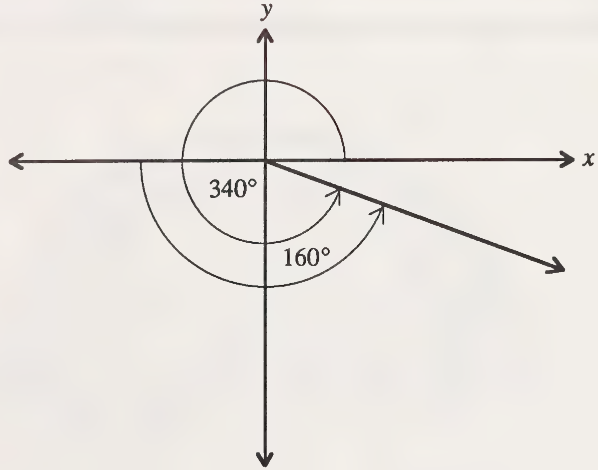
b. $310° - 180° = 130°$



c.



d.



3. a. 245° b. 320°

Extensions

1. a. $\frac{7}{36}\pi$ b. $-\frac{2}{3}\pi$

   c. $1\frac{7}{9}\pi$

2. a. 2520° b. −72°

   c. 144°

3. a. −1 b. −1

   c. −1

4. a. $\frac{-7}{30}\pi$ b. $\frac{23}{18}\pi$

   c. $\frac{20}{9}\pi$

5. a. 1980° b. 300°

   c. 77.14°

6. a. $\frac{1}{2}$ or 0.5 b. −0.7071

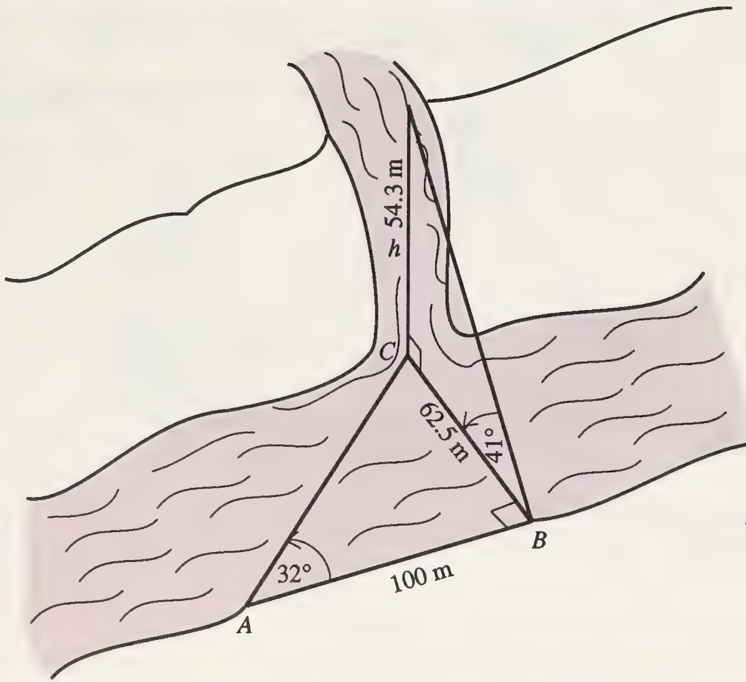
   c. −0.5774

Exploring Topic 3

Activity 1

Find the measures of sides and angles in multiple right triangles and solve problems involving multiple right triangles in two or three dimensions.

1.



$$\tan 32° = \frac{CB}{100\text{ m}}$$

$$CB = (100\text{ m})(\tan 32°)$$

$$\doteq 62.5\text{ m}$$

The width of the lava flow at ground level is about 62.5 m.

$$\tan 41° = \frac{h}{CB}$$

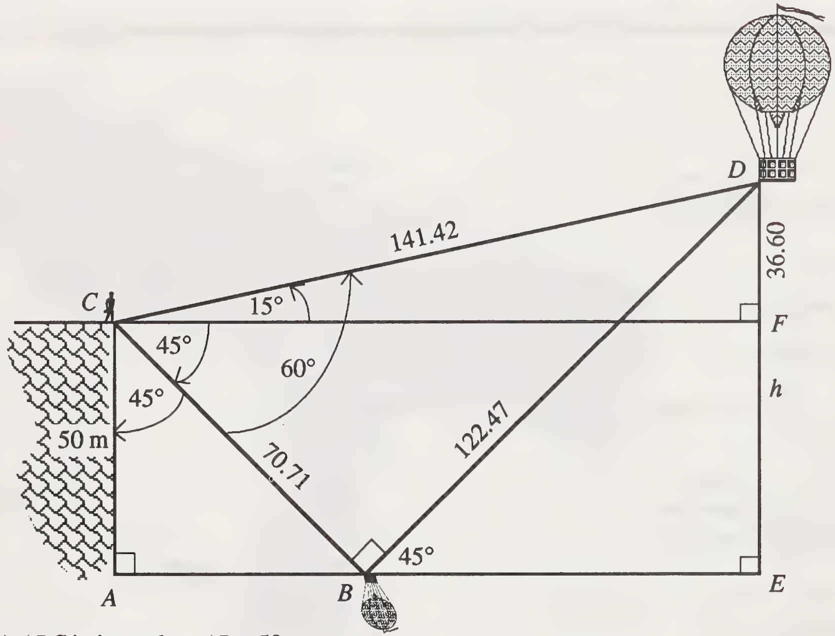
$$h = CB(\tan 41°)$$

$$\doteq 62.5\text{ m}(0.869\,286\,737)$$

$$\doteq 54.3\text{ m}$$

The height of the cliff is about 54.3 m.

2.



Since $\Delta\, ABC$ is isosceles, $AB = 50$ m.

$$CB = \sqrt{(50\text{ m})^2 + (50\text{ m})^2}$$

$$CB \doteq 70.71\text{ m}$$

$$\cos\ \angle BCD \doteq \frac{70.71\text{ m}}{CD}$$

$$CD \doteq \frac{70.71\text{ m}}{\cos\ 60°}$$

$$\doteq 141.42\text{ m}$$

$$\tan\ 60° \doteq \frac{BD}{70.71\text{ m}}$$

$$BD \doteq (70.71\text{ m})(\tan\ 60°)$$

$$\doteq 122.47\text{ m}$$

$$\sin\ \angle EBD = \frac{h}{BD}$$

$$h = BD(\sin\ 45°)$$

$$\doteq (122.47\text{ m})(0.707\,106\,781)$$

$$\doteq 86.60\text{ m}$$

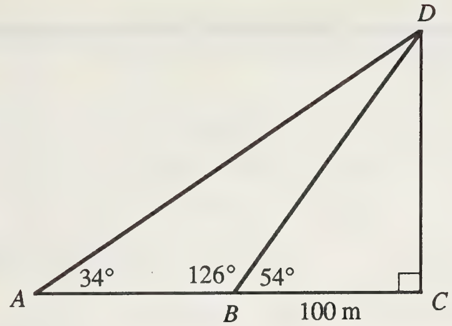
$$\sin\ 15° \doteq \frac{DF}{141.42\text{ m}}$$

$$DF \doteq (141.42\text{ m})(\sin\ 15°)$$

$$\doteq 36.60\text{ m}$$

Therefore, the balloon is approximately 86.60 m above the lake, and it is approximately 36.60 m above the hiker.

3.



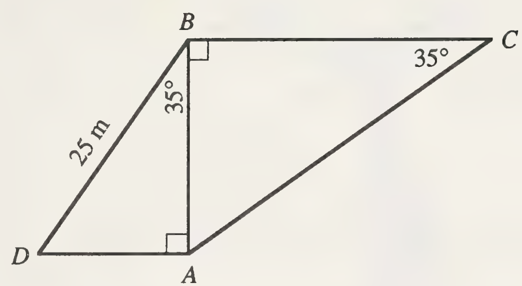
$$\tan 54° = \frac{DC}{BC}$$

$$DC = BC(\tan 54°)$$
$$\doteq (100 \text{ m})(1.376\,381\,92)$$
$$\doteq 137.6 \text{ m}$$

$$\sin 34° = \frac{DC}{AD}$$

$$AD = \frac{DC}{\sin 34°}$$
$$\doteq \frac{137.6 \text{ m}}{0.5592}$$
$$AD \doteq 246.1 \text{ m}$$

4.



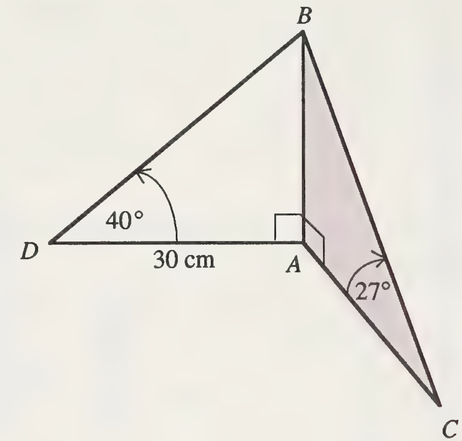
$$\cos 35° = \frac{AB}{25 \text{ m}}$$

$$AB = (25 \text{ m})(\cos 35°)$$
$$\doteq 20.48 \text{ m}$$

$$\sin 35° = \frac{AB}{AC}$$

$$AC = \frac{AB}{\sin 35°}$$
$$\doteq \frac{20.48 \text{ m}}{0.5736}$$
$$AC \doteq 35.71 \text{ m}$$

5.



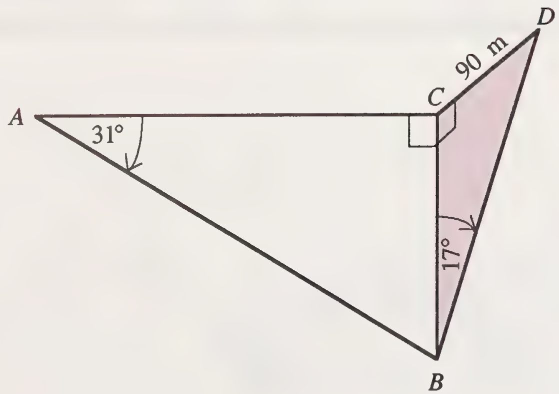
$$\tan 40° = \frac{AB}{AD}$$

$$AB = AD(\tan 40°)$$
$$\doteq (30 \text{ cm})(0.8391)$$
$$\doteq 25.17 \text{ cm}$$

$$\sin 27° \doteq \frac{AB}{BC}$$

$$BC \doteq \frac{AB}{\sin 27°}$$
$$\doteq \frac{25.17 \text{ cm}}{0.454}$$
$$BC \doteq 55.44 \text{ cm}$$

6.



$$\tan 17° = \frac{CD}{BC}$$

$$BC = \frac{CD}{\tan 17°}$$

$$\doteq \frac{90 \text{ m}}{0.305\,731}$$

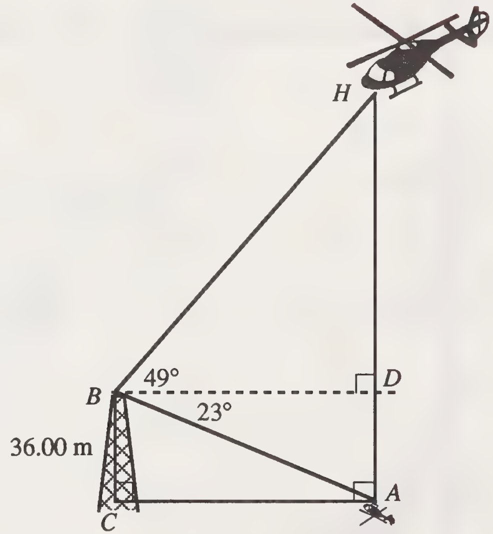
$$\doteq 294.38 \text{ m}$$

$$\sin 31° = \frac{BC}{AB}$$

$$AB \doteq \frac{294.38 \text{ m}}{0.5150}$$

$$\doteq 571.57 \text{ m}$$

7.



ADBC forms a rectangle; therefore, $\angle CBD$ is equal to 90°.

$$\angle CBD = \angle ABD + \angle CBA$$

$$90° = 23° + \angle CBA$$

$$\angle CBA = 67°$$

$$\tan 67° = \frac{CA}{36.00 \text{ m}}$$

$$CA = 36.00 \text{ m} \times \tan 67°$$

$$CA \doteq 36.00 \text{ m} \times 2.3559$$

$$CA \doteq 84.81 \text{ m}$$

The shadow is about 84.81 m from the base of the tower.

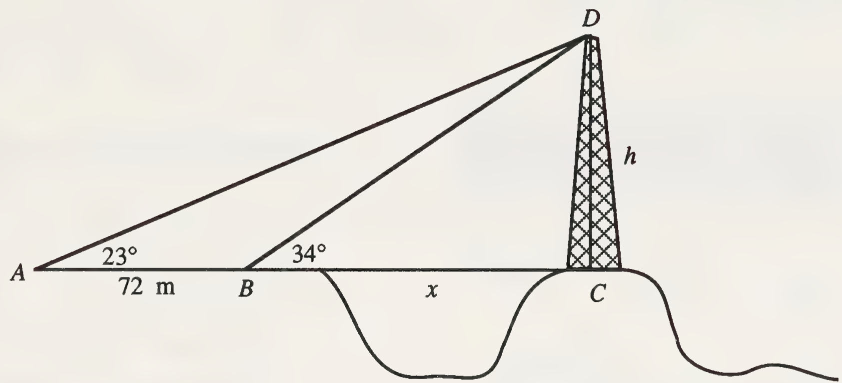
ADBC forms a rectangle.
Therefore, $CA = BD = 84.81\text{ m}$ and $BC = AD = 36.00\text{ m}$.

$$\tan 49° = \frac{DH}{84.81\text{ m}}$$
$$DH = 84.81\text{ m} \times \tan 49°$$
$$DH \doteq 84.81\text{ m} \times 1.1504$$
$$DH \doteq 97.56\text{ m}$$

$$AH = AD + DH$$
$$AH \doteq 36.00\text{ m} + 97.56\text{ m}$$
$$AH \doteq 133.56\text{ m}$$

The helicopter is about 133.56 m above the shadow.

8.



Note that $\tan 23°$ is equal to $\frac{h}{x + 72\text{ m}}$ and $\tan 34°$ is equal to $\frac{h}{x}$.

Solve both equations for h, and then solve for x.

$h = (\tan 23°)(x + 72 \text{ m})$ and $h = x(\tan 34°)$

$$\begin{aligned} h &= h \\ (\tan 23°)(x+72 \text{ m}) &= x(\tan 34°) \\ x \tan 23° + (72 \text{ m})(\tan 23°) &= x \tan 34° \\ x \tan 34° - x \tan 23° &= (72 \text{ m})(\tan 23°) \\ x(\tan 34° - \tan 23°) &= (72 \text{ m})(\tan 23°) \\ x &= \frac{(72 \text{ m})(\tan 23°)}{\tan 34° - \tan 23°} \\ x &\doteq 122.23 \text{ m} \end{aligned}$$

Calculate the height of the tower and the lengths of the guy wires.

$$\begin{aligned} \tan 34° &= \frac{h}{x} \\ h &= x \tan 34° \\ &\doteq (122.23 \text{ m})(\tan 34°) \\ &\doteq 82.45 \text{ m} \end{aligned}$$

The height of the tower is about 82.45 m.

$$\begin{aligned} BD &= \sqrt{x^2 + h^2} \\ &\doteq \sqrt{(122.23 \text{ m})^2 + (82.45 \text{ m})^2} \\ &\doteq 147.44 \text{ m} \end{aligned}$$

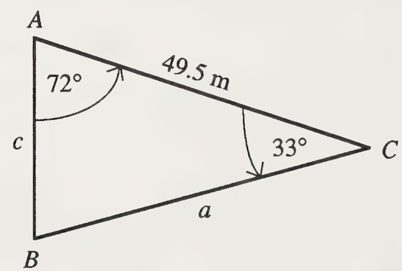
$$\begin{aligned} \sin 23° &= \frac{h}{AD} \\ AD &= \frac{h}{\sin 23°} \\ &\doteq \frac{82.45 \text{ m}}{\sin 23°} \\ &\doteq 211.01 \text{ m} \end{aligned}$$

The lengths of the guy wires are about 147.44 m and 211.01 m respectively.

Activity 2

Find the measures of unknown sides and angles in oblique triangles by applying the sine law.

1.



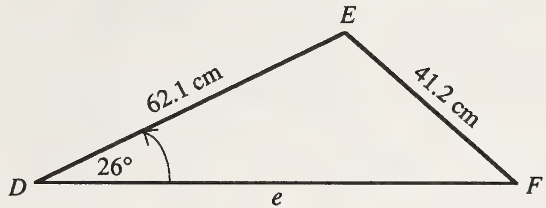
$\angle B = 180° - (72° + 33°)$

$\angle B = 75°$

$$\frac{a}{\sin A}=\frac{b}{\sin B}$$
$$a=\frac{b\sin A}{\sin B}$$
$$a=\frac{(49.5\text{ m})(\sin 72°)}{(\sin 75°)}$$
$$a \doteq 48.7\text{ m}$$

$$\frac{c}{\sin C}=\frac{b}{\sin B}$$
$$c=\frac{b\sin C}{\sin B}$$
$$c=\frac{(49.5\text{ m})(\sin 33°)}{\sin 75°}$$
$$c \doteq \frac{(49.5\text{ m})(0.5446)}{(0.9659)}$$
$$c \doteq 27.9\text{ m}$$

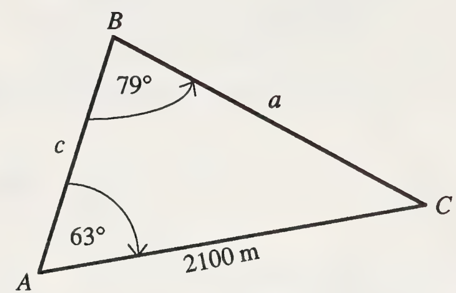
2.



$$\frac{f}{\sin F}=\frac{d}{\sin D}$$
$$\sin F=\frac{f\sin D}{d}$$
$$\sin F=\frac{(62.1\text{ cm})(\sin 26°)}{41.2\text{ cm}}$$
$$\sin F \doteq \frac{(62.1\text{ cm})(0.4384)}{41.2\text{ cm}}$$
$$\sin F \doteq 0.6607$$
$$\angle F \doteq 41°$$

$$\angle E \doteq 180°-(26°+41°)$$
$$\angle E \doteq 113°$$
$$\frac{e}{\sin E}=\frac{d}{\sin D}$$
$$e=\frac{d\sin E}{\sin D}$$
$$e \doteq \frac{(41.2\text{ cm})(\sin 113°)}{\sin 26°}$$
$$e \doteq \frac{(41.2\text{ cm})(0.9205)}{(0.4384)}$$
$$e \doteq 86.5\text{ cm}$$

3.

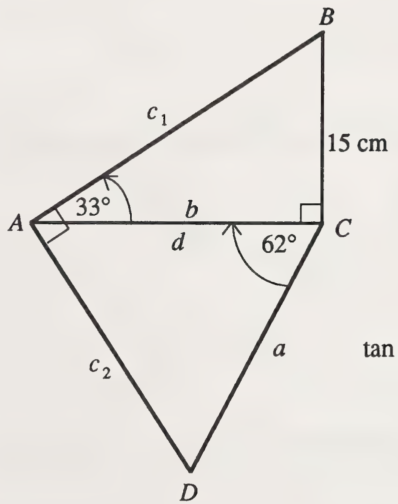


$$\angle C=180°-(63°+79°)$$
$$\angle C=38°$$

$$\frac{a}{\sin A}=\frac{b}{\sin B}$$
$$a=\frac{b\sin A}{\sin B}$$
$$a=\frac{(2100\text{ m})(\sin 63^\circ)}{\sin 79^\circ}$$
$$a\doteq\frac{(2100\text{ m})(0.8910)}{0.9816}$$
$$a\doteq 1906\text{ m}$$

$$\frac{c}{\sin C}=\frac{b}{\sin B}$$
$$c=\frac{b\sin C}{\sin B}$$
$$c=\frac{(2100\text{ m})(\sin 38^\circ)}{\sin 79^\circ}$$
$$c\doteq\frac{(2100\text{ m})(0.6157)}{0.9816}$$
$$c\doteq 1317\text{ m}$$

4.



$$\tan 33^\circ=\frac{15\text{ cm}}{b}$$
$$b=\frac{15\text{ cm}}{\tan 33^\circ}$$
$$b\doteq\frac{15\text{ cm}}{0.6494}$$
$$b\doteq 23\text{ cm}$$
$$b=d$$
$$\therefore d\doteq 23\text{ cm}$$

$$\sin 33^\circ=\frac{15\text{ cm}}{c_1}$$
$$c_1=\frac{15\text{ cm}}{\sin 33^\circ}$$
$$c_1\doteq\frac{15\text{ cm}}{0.5446}$$
$$c_1\doteq 27.5\text{ cm}$$

$$\angle B=90^\circ-33^\circ$$
$$\angle B=57^\circ$$

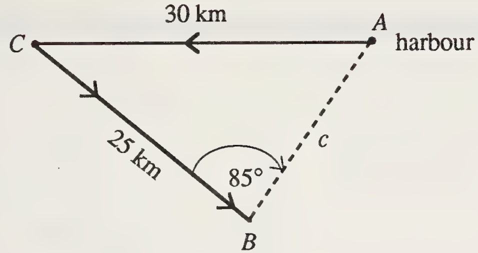
$$\angle CAD=90^\circ-33^\circ$$
$$\angle CAD=57^\circ$$

$$\angle D=180^\circ-(62^\circ+57^\circ)$$
$$\angle D=61^\circ$$

$$\frac{d}{\sin 61^\circ}=\frac{a}{\sin\angle CAD}$$
$$a=\frac{d\sin\angle CAD}{\sin 61^\circ}$$
$$a\doteq\frac{(23\text{ cm})(\sin 57^\circ)}{\sin 61^\circ}$$
$$a\doteq 22\text{ cm}$$

$$\frac{c_2}{\sin 62^\circ}=\frac{a}{\sin 57^\circ}$$
$$c_2=\frac{a\sin 62^\circ}{\sin 57^\circ}$$
$$c_2\doteq\frac{(22\text{ cm})(0.8829)}{(0.8387)}$$
$$c_2\doteq 23\text{ cm}$$

5.



$$\frac{25\text{ km}}{\sin A} = \frac{30\text{ km}}{\sin 85°}$$

$$\sin A = \frac{(25\text{ km})(\sin 85°)}{30\text{ km}}$$

$$\sin A \doteq 0.8302$$

$$\angle A \doteq 56°$$

$$\angle C \doteq 180° - (85° + 56°)$$

$$\angle C \doteq 39°$$

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$c = \frac{a \sin C}{\sin A}$$

$$c = \frac{(25\text{ km})(\sin 39°)}{\sin 56°}$$

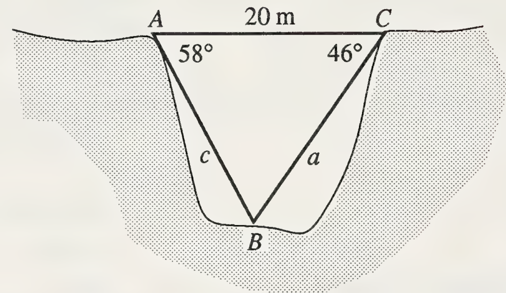
$$c \doteq \frac{(25\text{ km})(0.6293)}{(0.8290)}$$

$$c \doteq 19\text{ km}$$

The ship is approximately 19 km from the harbour when it is at point B.

6. Let a be the length of the support from B to C and let c be the length of the support from B to A.

Now, $\angle B = 180° - (58° + 46°)$
$= 180° - 104°$
$= 76°$



Using the sine law, you obtain the following:

$$\frac{a}{\sin 58°} = \frac{20}{\sin 76°} = \frac{c}{\sin 46°}$$

Then, $a = \dfrac{20(\sin 58°)}{\sin 76°}$
$\doteq \dfrac{20(0.8480)}{0.9703}$
$\doteq 17.4791$
$\doteq 17.5$

$c = \dfrac{20(\sin 46°)}{\sin 76°}$
$\doteq \dfrac{20(0.7193)}{0.9703}$
$\doteq 14.8263$
$\doteq 14.8$

Therefore, the lengths of the supports for the bridge will be approximately 17.5 m and 14.8 m respectively.

Activity 3

Find the measures of unknown sides and angles in oblique triangle by applying the cosine law.

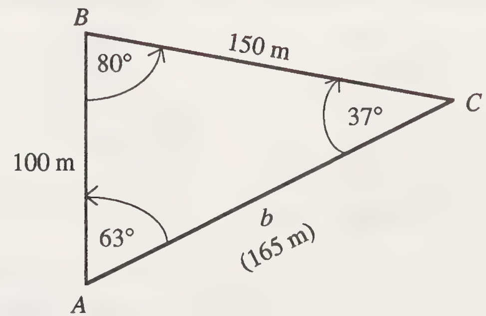
1. Use the cosine law for the following:

$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$b^2 = (150\text{ m})^2 + (100\text{ m})^2 - 2(150\text{ m})(100\text{ m})\cos 80°$$
$$b^2 = 22\,500\text{ m}^2 + 10\,000\text{ m}^2 - 30\,000\text{ m}^2(\cos 80°)$$
$$b^2 \doteq 27\,290.55\text{ m}^2$$
$$b \doteq 165\text{ m}$$

Solve for the other angles using the sine law or the cosine law. The solution that follows uses the cosine law.

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$\cos A = \frac{a^2 - b^2 - c^2}{-2bc}$$
$$\cos A = \frac{(150\text{ m})^2 - (165\text{ m})^2 - (100\text{ m})^2}{-2(165\text{ m})(100\text{ m})}$$
$$\cos A \doteq 0.4462$$
$$\angle A \doteq 63°$$

$$\angle C \doteq 180° - (80° + 63°)$$
$$\angle C \doteq 37°$$

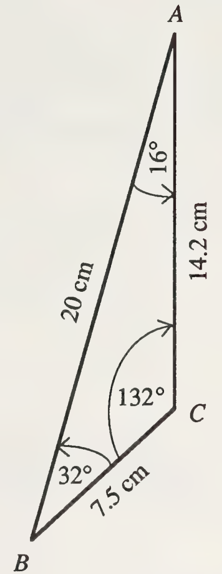


2. $b^2 = a^2 + c^2 - 2ac \cos B$

$$b^2 = (7.5\text{ cm})^2 + (20\text{ cm})^2 - 2(7.5\text{ cm})(20\text{ cm})\cos 32°$$
$$b^2 \doteq 201.8\text{ cm}^2$$
$$b \doteq 14.2\text{ cm}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$\cos A = \frac{a^2 - b^2 - c^2}{-2bc}$$
$$\cos A = \frac{(7.5\text{ cm})^2 - (14.2\text{ cm})^2 - (20\text{ cm})^2}{-2(14.2\text{ cm})(20\text{ cm})}$$
$$\cos A \doteq 0.9602$$
$$\angle A \doteq 16°$$

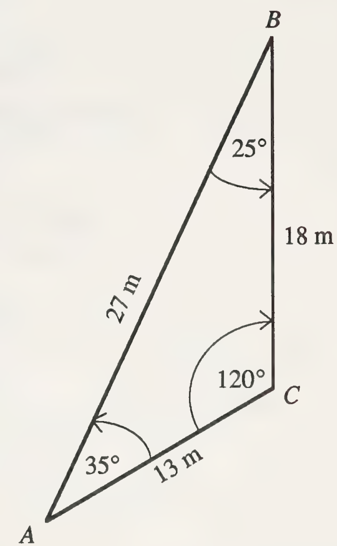
$$\angle C \doteq 180° - (32° + 16°)$$
$$\angle C \doteq 132°$$



3. $$a^2 = b^2 + c^2 - 2bc \cos A$$
$$\cos A = \frac{a^2 - b^2 - c^2}{-2bc}$$
$$\cos A = \frac{(18 \text{ m})^2 - (13 \text{ m})^2 - (27 \text{ m})^2}{-2(13 \text{ m})(27 \text{ m})}$$
$$\cos A \doteq 0.8177$$
$$\angle A \doteq 35°$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$\cos B = \frac{b^2 - a^2 - c^2}{-2ac}$$
$$\cos B = \frac{(13 \text{ m})^2 - (18 \text{ m})^2 - (27 \text{ m})^2}{-2(18 \text{ m})(27 \text{ m})}$$
$$\cos B \doteq 0.9095$$
$$\angle B \doteq 25°$$

$$\angle C \doteq 180° - (35° + 25°)$$
$$\angle C \doteq 120°$$



4. $a^2 = b^2 + c^2 - 2bc \cos A$

$$\cos A = \frac{a^2 - b^2 - c^2}{-2bc}$$

$$\cos A = \frac{(64)^2 - (75)^2 - (135)^2}{-2(75)(135)}$$

$\cos A \doteq 0.9755$

$\angle A \doteq 13°$

$b^2 = a^2 + c^2 - 2ac \cos B$

$$\cos B = \frac{b^2 - a^2 - c^2}{-2ac}$$

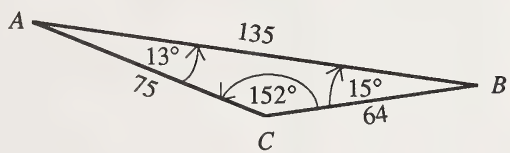
$$\cos B = \frac{(75)^2 - (64)^2 - (135)^2}{-2(64)(135)}$$

$\cos B \doteq 0.9662$

$\angle B \doteq 15°$

$\angle C \doteq 180° - (13° + 15°)$

$\angle C \doteq 152°$



5. First deal with the right-angled triangle.

$$\sin 40° = \frac{d}{9}$$

$d = 9 \times \sin 40°$

$d \doteq 9 \times 0.6428$

$d \doteq 5.8$

$$\cos 40° = \frac{BD}{9}$$

$BD = 9 \times \cos 40°$

$BD \doteq 9 \times 0.7660$

$BD \doteq 6.9$

$\angle BCD = 180° - (40° + 90°)$

$\angle BCD = 180° - 130°$

$\angle BCD = 50°$

$d \doteq a \doteq 5.8$

$a^2 = b^2 + c^2 - 2bc \cos A$

$$\cos A = \frac{a^2 - b^2 - c^2}{-2bc}$$

$$\cos A \doteq \frac{5.8^2 - 16^2 - 11.5^2}{-2 \times 16 \times 11.5}$$

$\cos A \doteq 0.9636$

$\angle A \doteq 16°$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$\cos B \doteq \frac{b^2 - a^2 - c^2}{-2ac}$$

$$\cos B \doteq \frac{16^2 - 5.8^2 - 11.5^2}{-2\times 5.8\times 11.5}$$

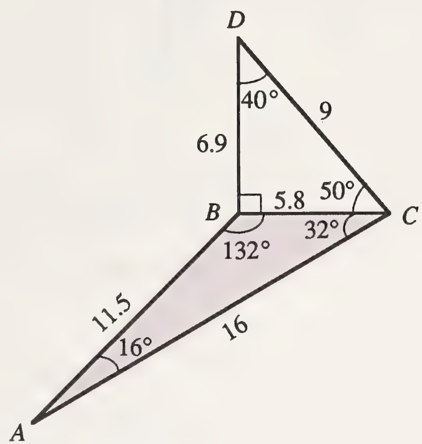
$$\cos B \doteq -0.6755$$

$$\angle B \doteq 132°$$

$$\angle ACB \doteq 180° - (16° + 132°)$$

$$\angle ACB \doteq 180° - 148°$$

$$\angle ACB \doteq 32°$$



6. $\angle B = 180° - (90° + 27°)$

$\angle B = 180° - 117°$

$\angle B = 63°$

$$\sin 27° = \frac{24.5}{AB}$$

$$AB = \frac{24.5}{\sin 27°}$$

$$AB \doteq \frac{24.5}{0.4540}$$

$$AB \doteq 54.0$$

$$\tan 27° = \frac{24.5}{b}$$

$$b = \frac{24.5}{\tan 27°}$$

$$b \doteq \frac{24.5}{0.5095}$$

$$b \doteq 48.1$$

$$b \doteq d \doteq 48.1$$

$$c^2 = a^2 + d^2 - 2ad\cos C$$

$$c^2 \doteq 46^2 + 48.1^2 - 2\times 46\times 48.1\times\cos 120°$$

$$c^2 \doteq 2116 + 2313.61 - 4425.2(-0.5)$$

$$c^2 \doteq 6642.21$$

$c \doteq 81.5$ (This is side AD.)

$$d^2 = a^2 + c^2 - 2ac\cos D$$

$$\cos D = \frac{d^2 - a^2 - c^2}{-2ac}$$

$$\cos D \doteq \frac{48.1^2 - 46^2 - 81.5^2}{-2 \times 46 \times 81.5}$$

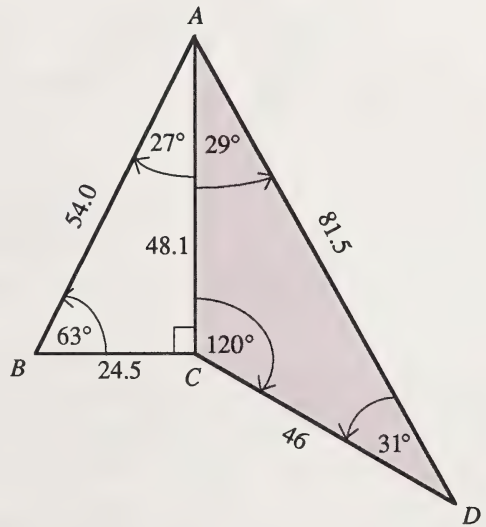
$$\cos D \doteq 0.8595$$

$$\angle D \doteq 31°$$

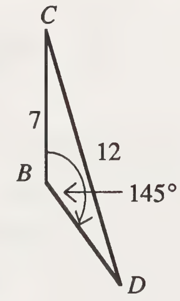
$$\angle CAD \doteq 180° - (120° + 31°)$$

$$\angle CAD \doteq 180° - 151°$$

$$\angle CAD \doteq 29°$$



7. First solve ΔBCD using the sine law.



$$\frac{b}{\sin B} = \frac{d}{\sin D}$$

$$\sin D = \frac{d \sin B}{b}$$

$$\sin D = \frac{7(\sin 145°)}{12}$$

$$\sin D \doteq 0.3346$$

$$\angle D \doteq 20°$$

$$\angle C \doteq 180° - (145° + 20°)$$

$$\angle C \doteq 15°$$

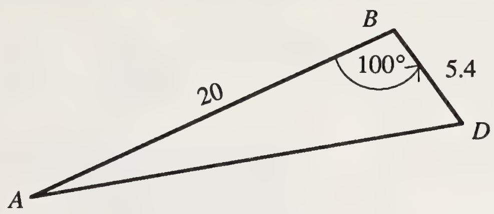
$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

$$c = \frac{b \sin C}{\sin B}$$

$$c \doteq \frac{12(\sin 15°)}{\sin 145°}$$

$$c \doteq 5.4$$

Solve the second triangle using the cosine law.



$b^2 = a^2 + d^2 - 2ad \cos B$

$b^2 \doteq (5.4)^2 + (20)^2 - 2(5.4)(20) \cos 100°$

$b^2 \doteq 466.7$

$b \doteq 21.6$

$a^2 = b^2 + d^2 - 2bd \cos A$

$\cos A = \dfrac{a^2 - b^2 - d^2}{-2bd}$

$\cos A \doteq \dfrac{(5.4)^2 - (21.6)^2 - (20)^2}{-2(21.6)(20)}$

$\cos A \doteq 0.9692$

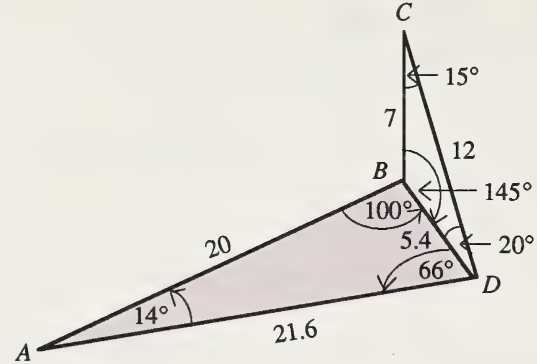
$\angle A \doteq 14°$

$\angle D \doteq 180° - (100° + 14°)$

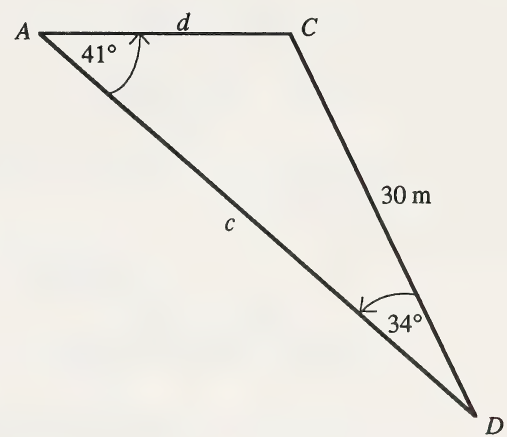
$\angle D \doteq 180° - 114°$

$\angle D \doteq 66°$

The complete solution is as follows:



8. Begin by solving ΔACD using the sine law.



$$\frac{a}{\sin A}=\frac{d}{\sin D}$$

$$d=\frac{a\sin D}{\sin A}$$

$$d=\frac{(30\text{ m})(\sin 34°)}{\sin 41°}$$

$$d \doteq 25.6\text{ m}$$

$$\angle C=180°-(34°+41°)$$

$$\angle C=180°-75°$$

$$\angle C=105°$$

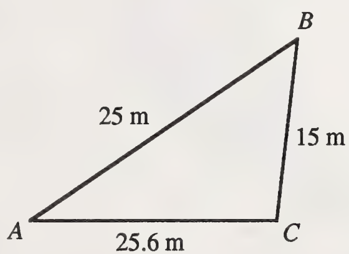
$$\frac{c}{\sin C}=\frac{a}{\sin A}$$

$$c=\frac{a\sin C}{\sin A}$$

$$c=\frac{(30\text{ m})(\sin 105°)}{\sin 41°}$$

$$c \doteq 44.2\text{ m}$$

Now solve ΔABC.



$$a^2=b^2+c^2-2bc\cos A$$

$$\cos A=\frac{a^2-b^2-c^2}{-2bc}$$

$$\cos A \doteq \frac{(15\text{ m})^2-(25.6\text{ m})^2-(25\text{ m})^2}{-2(25.6\text{ m})(25\text{ m})}$$

$$\cos A \doteq 0.8245$$

$$\angle A \doteq 34°$$

$$b^2=a^2+c^2-2ac\cos B$$

$$\cos B=\frac{b^2-a^2-c^2}{-2ac}$$

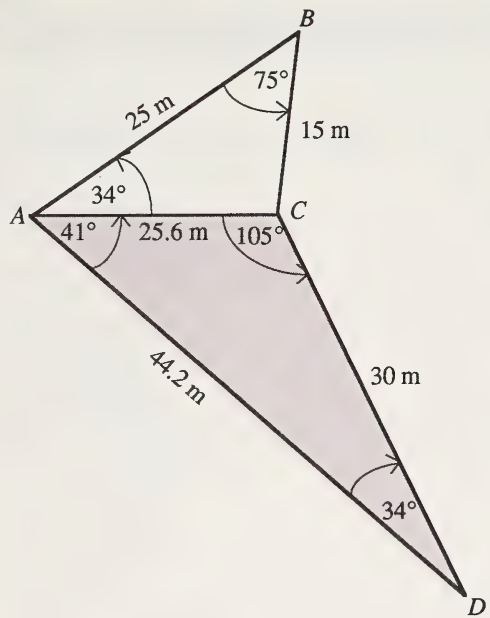
$$\cos B \doteq \frac{(25.6\text{ m})^2-(15\text{ m})^2-(25\text{ m})^2}{-2(15\text{ m})(25\text{ m})}$$

$$\cos B \doteq 0.259\,52$$

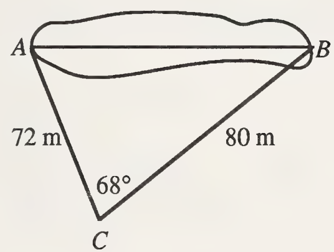
$$\angle B \doteq 75°$$

$$\angle C \doteq 180°-(75°+34°)$$

$$\angle C \doteq 71°$$



9.



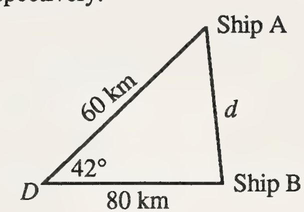
Using the cosine law, you get the following:

$$
\begin{aligned}
AB^2 &= 72^2 + 80^2 - 2(72)(80)\cos 68° \\
&\doteq 5184 + 6400 - 11\,520(0.3746) \\
&\doteq 7268.608
\end{aligned}
$$

$$
\begin{aligned}
\text{Then, } AB &\doteq \sqrt{7268.608} \\
&\doteq 85.2561 \\
&\doteq 85.3
\end{aligned}
$$

Therefore, the distance across the pond is about 85.3 m.

10. Let d be the distance between the ships after four hours. The distances travelled by the ships in four hours will be 60 km and 80 km, respectively.



$$
\begin{aligned}
\text{Then, } d^2 &= 60^2 + 80^2 - 2(60)(80)\cos 42° \\
&\doteq 3600 + 6400 - 9600(0.7431) \\
&\doteq 2866.24 \\
d &\doteq \sqrt{2866.24} \\
&\doteq 53.5372 \\
&\doteq 53.5
\end{aligned}
$$

After four hours the ships will be 53.5 km apart.

Extra Help

1. a. SAS requires the cosine law.

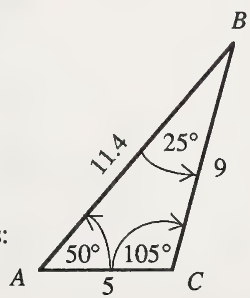
$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$c^2 = 9^2 + 5^2 - 2(9)(5)\cos 105°$$
$$c^2 \doteq 81 + 25 - 90(-0.2588)$$
$$c^2 \doteq 129.3$$
$$c \doteq 11.4$$

Use either the cosine law or the sine law to solve for one angle.

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$
$$\sin A = \frac{a \sin C}{c}$$
$$\doteq \frac{9(\sin 105°)}{11.4}$$
$$\doteq 0.7626$$
$$\angle A \doteq 50°$$

$$\therefore \angle B \doteq 180° - (50° + 105°)$$
$$\doteq 180° - 155°$$
$$\doteq 25°$$

The completed diagram is as follows:

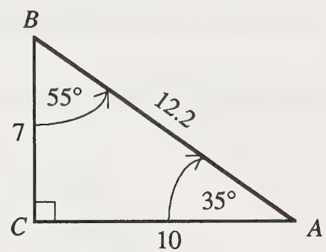


b. This is a right-angled triangle.

$$\tan A = \frac{7}{10}$$
$$\tan A = 0.7$$
$$\angle A \doteq 35°$$

$$\angle B \doteq 180° - (90° + 35°)$$
$$\angle B \doteq 180° - 125°$$
$$\angle B \doteq 55°$$

$$c^2 = a^2 + b^2$$
$$c^2 = 7^2 + 10^2$$
$$c^2 = 49 + 100$$
$$c^2 = 149$$
$$c \doteq 12.2$$



c. This is an SSA situation which can be solved by using the sine law. $\angle C_r$ is the angle related to $\angle C$.

$$\frac{b}{\sin B}=\frac{c}{\sin C_r}$$

$$\sin C_r=\frac{c\sin B}{b}$$

$$\sin C_r=\frac{12(\sin 20°)}{6}$$

$$\sin C_r \doteq 0.6840$$

$$\angle C_r \doteq 43°$$

This is the related angle since $\angle C$ is greater than 90°.

$$\angle C \doteq 180° - 43°$$
$$\doteq 137°$$

$$\angle A \doteq 180° - (137° + 20°)$$
$$\angle A \doteq 23°$$

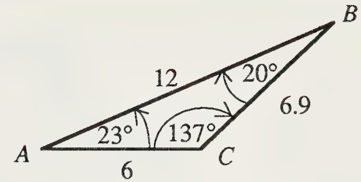
$$\frac{a}{\sin A}=\frac{b}{\sin B}$$

$$a=\frac{b\sin A}{\sin B}$$

$$a\doteq\frac{6(\sin 23°)}{\sin 20°}$$

$$a\doteq 6.9$$

The completed diagram is as follows:



d. This is an SAA situation which can be solved by using the sine law.

$$\frac{a}{\sin A}=\frac{b}{\sin B}$$

$$b=\frac{a\sin B}{\sin A}$$

$$b=\frac{15(\sin 120°)}{\sin 40°}$$

$$b\doteq 20.2$$

$$\angle C = 180° - (120° + 40°)$$
$$\angle C = 20°$$

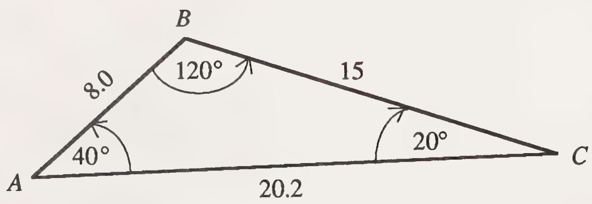
$$\frac{c}{\sin C}=\frac{b}{\sin B}$$

$$c=\frac{b\sin C}{\sin B}$$

$$c\doteq\frac{20.2(\sin 20°)}{\sin(120°)}$$

$$c\doteq 8.0$$

The completed diagram is as follows:



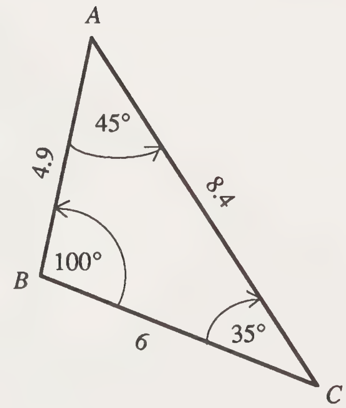
e. This is an ASA triangle which can be solved using the sine law.

$$\angle A = 180° - (100° + 35°)$$
$$\angle A = 45°$$

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$
$$b = \frac{a \sin B}{\sin A}$$
$$b = \frac{6(\sin 100°)}{\sin 45°}$$
$$b \doteq 8.4$$

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$
$$c = \frac{a \sin C}{\sin A}$$
$$c = \frac{6(\sin 35°)}{\sin 45°}$$
$$c \doteq 4.9$$

The completed diagram is as follows:



f. This is an SSS triangle which can be solved by using the cosine law.

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$\cos A = \frac{a^2 - b^2 - c^2}{-2bc}$$
$$\cos A = \frac{10^2 - 15^2 - 20^2}{-2(15)(20)}$$
$$\cos A = 0.875$$
$$\angle A \doteq 29°$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$\cos B = \frac{b^2 - a^2 - c^2}{-2ac}$$

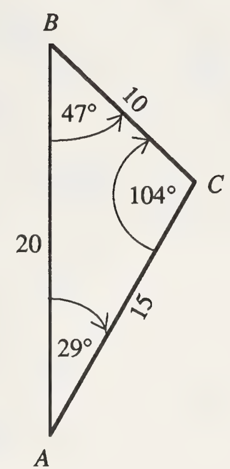
$$\cos B = \frac{15^2 - 10^2 - 20^2}{-2(10)(20)}$$

$$\cos B = 0.6875$$

$$\angle B \doteq 47°$$

$$\angle C \doteq 180° - (47° + 29°)$$

$$\angle C \doteq 104°$$



2. a. This is an SSA triangle. Use the sine law to solve the triangle.

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\sin B = \frac{b \sin C}{c}$$

$$\sin B = \frac{18(\sin 29°)}{10}$$

$$\sin B \doteq 0.8727$$

$$\angle B \doteq 61°$$

$$\angle A \doteq 180° - (61° + 29°)$$

$$\angle A \doteq 180° - 90°$$

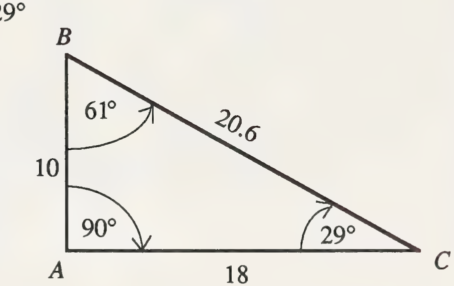
$$\angle A \doteq 90°$$

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$a = \frac{c \sin A}{\sin C}$$

$$a \doteq \frac{10(\sin 90°)}{\sin 29°}$$

$$a \doteq 20.6$$



b. This is an SSS triangle which can be solvedby using the cosine law.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\cos A = \frac{a^2 - b^2 - c^2}{-2bc}$$

$$\cos A = \frac{11^2 - 4^2 - 9^2}{-2(4)(9)}$$

$$\cos A \doteq -0.3333$$

$$\angle A \doteq 109°$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$\cos B = \frac{b^2 - a^2 - c^2}{-2ac}$$

$$\cos B = \frac{4^2 - 11^2 - 9^2}{-2(11)(9)}$$

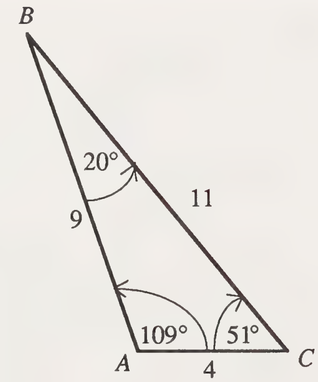
$$\cos B \doteq 0.9394$$

$$\angle B \doteq 20°$$

$$\angle C \doteq 180° - (20° + 109°)$$

$$\angle C \doteq 51°$$

The completed triangle is as follows:



c. This is an ASA triangle which can be solved by using the sine law.

$$\angle B = 180° - (70° + 30°)$$

$$\angle B = 180° - 100°$$

$$\angle B = 80°$$

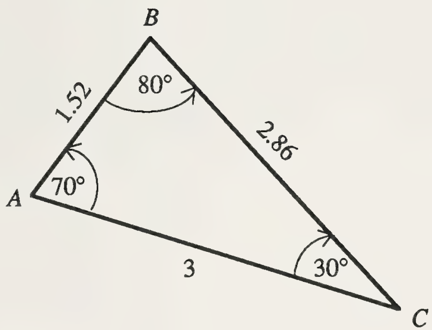
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$a = \frac{b \sin A}{\sin B}$$

$$a = \frac{3(\sin 70°)}{\sin 80°}$$

$$a \doteq 2.86$$

$$\frac{c}{\sin C}=\frac{b}{\sin B}$$
$$c=\frac{b\sin C}{\sin B}$$
$$c=\frac{3(\sin 30°)}{\sin 80°}$$
$$c \doteq 1.52$$

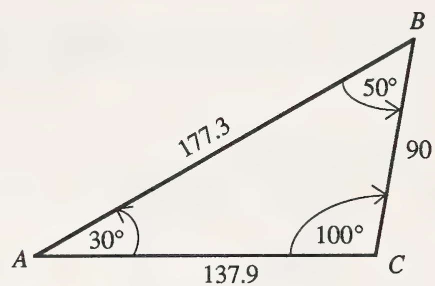


d. This is an SAA triangle which can be solved by using the sine law.

$$\angle C=180°-(50°+30°)$$
$$\angle C=180°-80°$$
$$\angle C=100°$$

$$\frac{b}{\sin B}=\frac{a}{\sin A}$$
$$b=\frac{a\sin B}{\sin A}$$
$$b=\frac{90(\sin 50°)}{\sin 30°}$$
$$b \doteq 137.9$$

$$\frac{c}{\sin C}=\frac{a}{\sin A}$$
$$c=\frac{a\sin C}{\sin A}$$
$$c=\frac{90(\sin 100°)}{\sin 30°}$$
$$c \doteq 177.3$$



e. This is an SAS triangle which requires the use of the cosine law.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$a^2 = 21^2 + 14^2 - 2(21)(14) \cos 74^\circ$$

$$a^2 \doteq 475$$

$$a \doteq 21.8$$

Use the sine law.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\sin B = \frac{b \sin A}{a}$$

$$\sin B \doteq \frac{21(\sin 74^\circ)}{21.8}$$

$$\sin B \doteq 0.9260$$

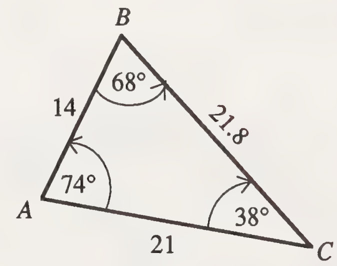
$$\angle B \doteq 68^\circ$$

$$\angle C \doteq 180^\circ - (68^\circ + 74^\circ)$$

$$\angle C \doteq 180^\circ - 142^\circ$$

$$\angle C \doteq 38^\circ$$

The completed diagram is as follows:



f. This is a right-angled triangle.

$$\cos 25^\circ = \frac{b}{16}$$

$$b = 16(\cos 25^\circ)$$

$$b \doteq 14.5$$

$$\sin 25^\circ = \frac{a}{16}$$

$$a = 16(\sin 25^\circ)$$

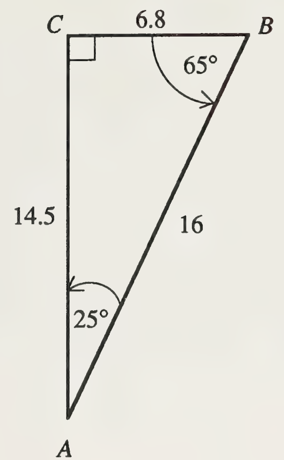
$$a \doteq 6.8$$

$$\angle B = 180^\circ - (90^\circ + 25^\circ)$$

$$\angle B = 180^\circ - 115^\circ$$

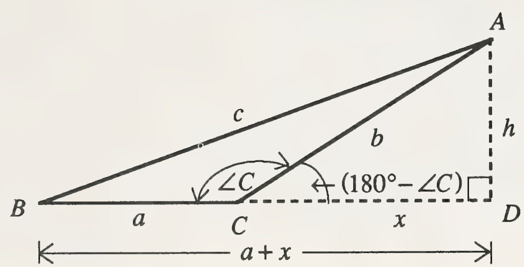
$$\angle B = 65^\circ$$

The completed diagram is as follows:



Extensions

ΔABC is an obtuse-angled triangle.



Draw AD perpendicular to BC.

In ΔACD,

$$\angle ACD = (180° - \angle C)$$

$$\cos \angle ACD = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{x}{b}$$

$$\therefore \cos(180° - \angle C) = \frac{x}{b}$$

But, $\cos(180° - \angle C) = -\cos C$.

Thus, $-\cos C = \dfrac{x}{b}$

and $x = -b\cos C$.

In ΔABE,

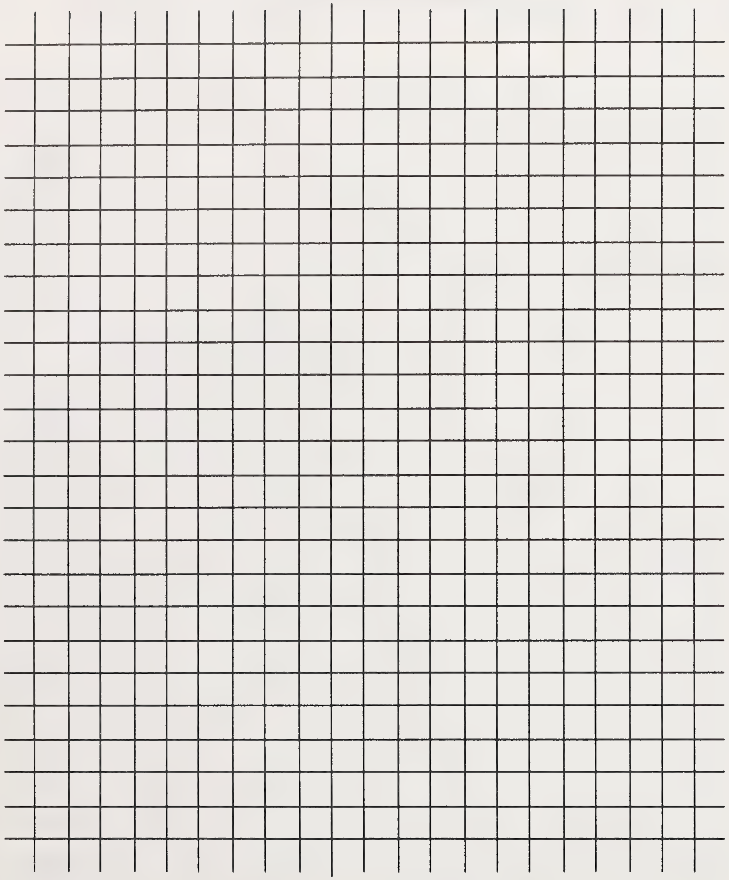
$$c^2 = h^2 + (a+x)^2$$
$$= h^2 + a^2 + 2ax + x^2$$
$$= a^2 + h^2 + x^2 + 2ax$$

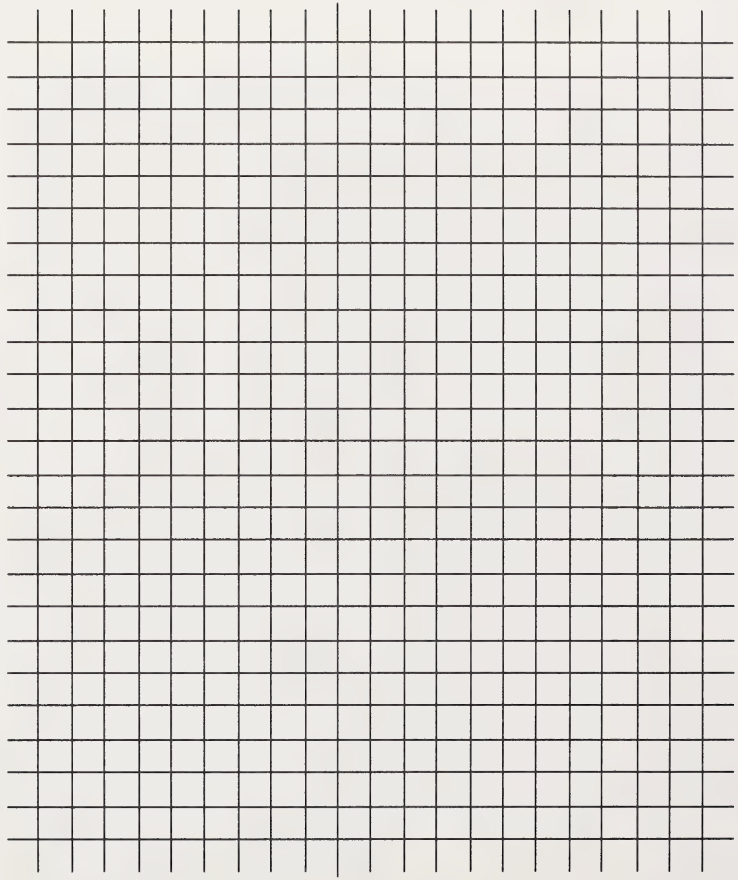
Substituting $(-b\cos C)$ for x and b^2 for $(h^2 + x^2)$, you get the following:

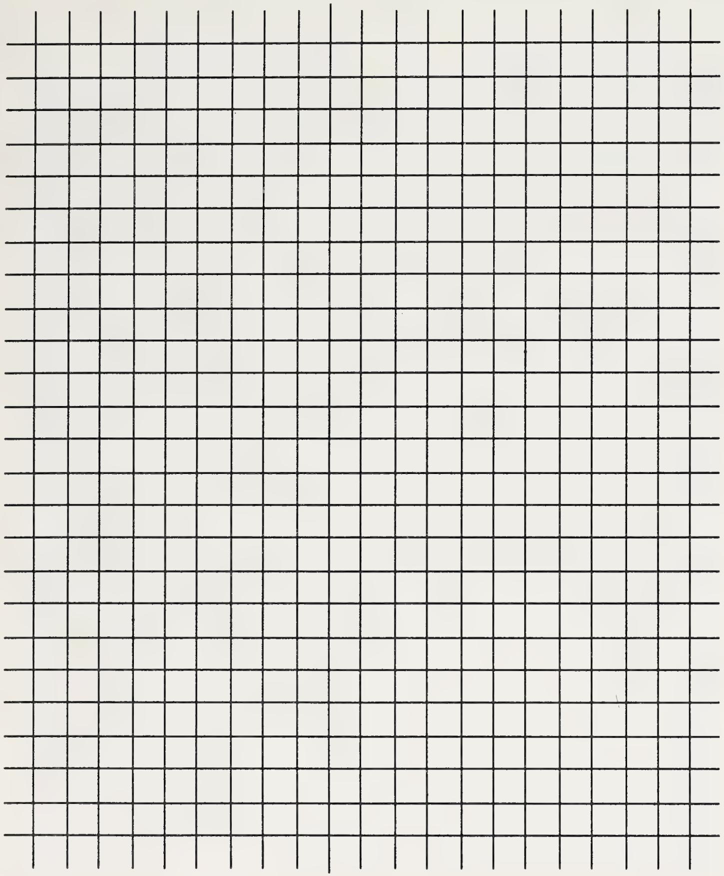
$$c^2 = a^2 + b^2 + 2a(-b\cos C)$$
$$\therefore c^2 = a^2 + b^2 - 2ab(\cos C)$$

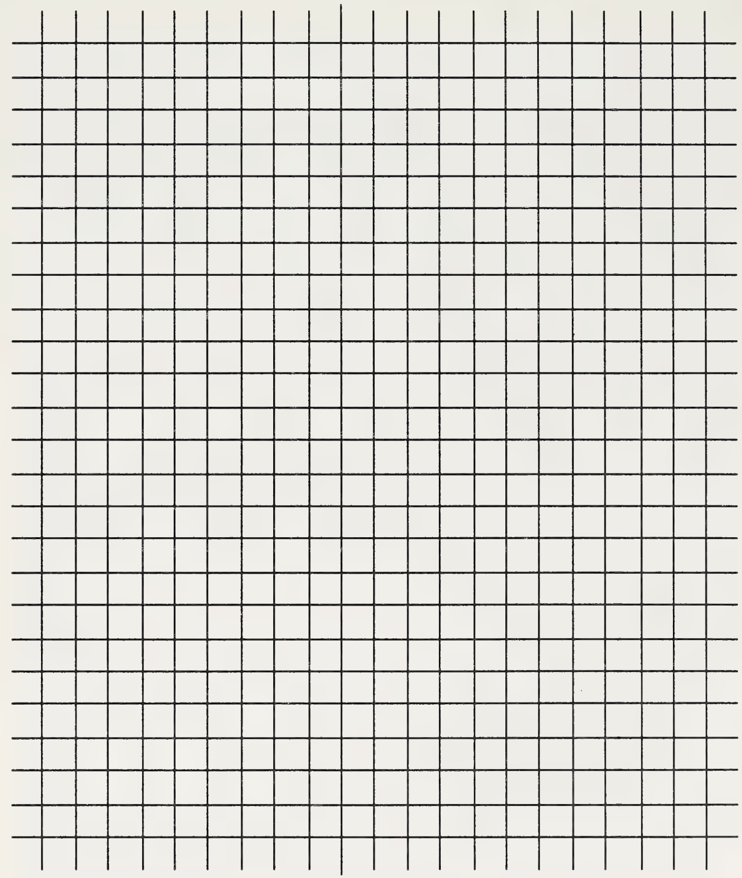
Appendix B
Graphing Material and Table

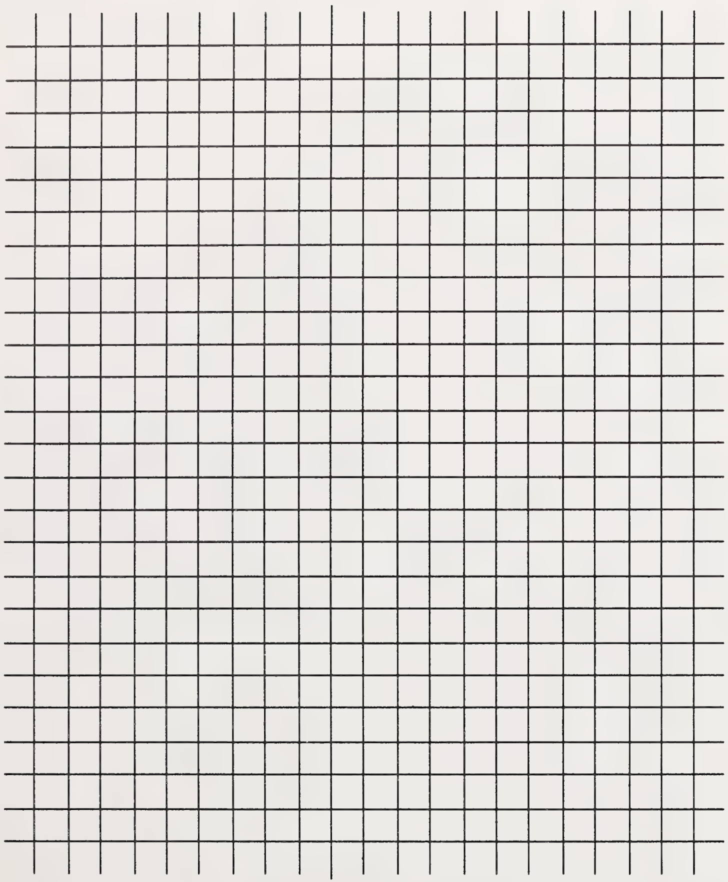
Graph Paper











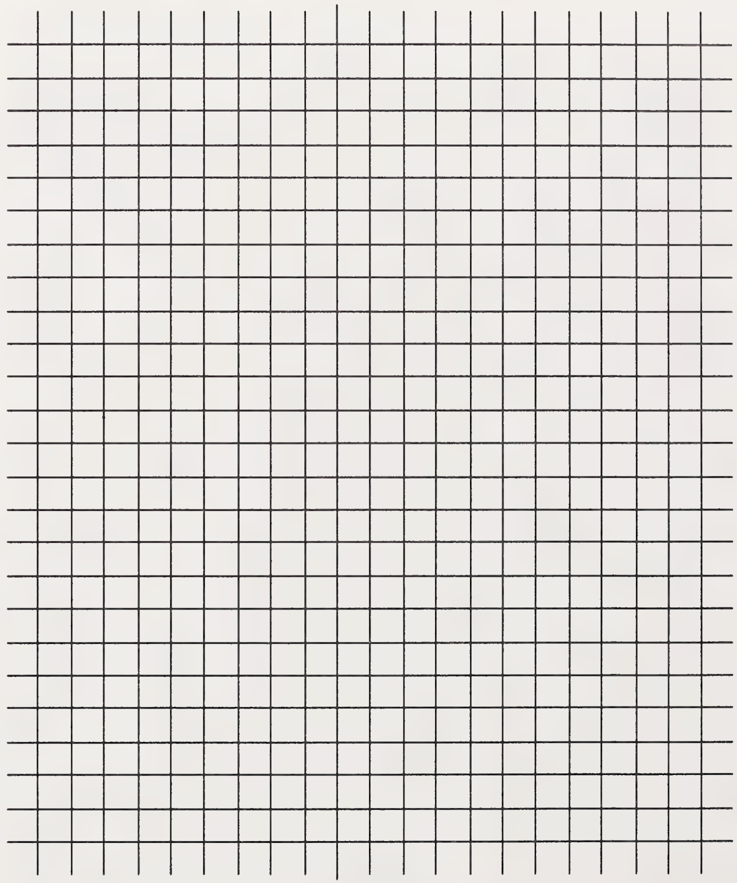


TABLE OF TRIGONOMETRIC FUNCTIONS

ANGLE deg	sin	cos	tan	csc	sec	cot
0	0.0000	1.0000	0.0000	undefined	1.0000	undefined
1	0.0175	0.9998	0.0175	57.2987	1.0002	57.2900
2	0.0349	0.9994	0.0349	28.6537	1.0006	28.6363
3	0.0523	0.9986	0.0524	19.1073	1.0014	19.0811
4	0.0698	0.9976	0.0699	14.3356	1.0024	14.3007
5	0.0872	0.9962	0.0875	11.4737	1.0038	11.4301
6	0.1045	0.9945	0.1051	9.5668	1.0055	9.5144
7	0.1219	0.9925	0.1228	8.2055	1.0075	8.1443
8	0.1392	0.9903	0.1405	7.1853	1.0098	7.1154
9	0.1564	0.9877	0.1584	6.3925	1.0125	6.3138
10	0.1736	0.9848	0.1763	5.7588	1.0154	5.6713
11	0.1908	0.9816	0.1944	5.2408	1.0187	5.1446
12	0.2079	0.9781	0.2126	4.8097	1.0223	4.7046
13	0.2250	0.9744	0.2309	4.4454	1.0263	4.3315
14	0.2419	0.9703	0.2493	4.1336	1.0306	4.0108
15	0.2588	0.9659	0.2679	3.8637	1.0353	3.7321
16	0.2756	0.9613	0.2867	3.6280	1.0403	3.4874
17	0.2924	0.9563	0.3057	3.4208	1.0457	3.2709
18	0.3090	0.9511	0.3249	3.2361	1.0515	3.0777
19	0.3256	0.9455	0.3443	3.0716	1.0576	2.9042
20	0.3420	0.9397	0.3640	2.9238	1.0642	2.7475
21	0.3584	0.9336	0.3839	2.7904	1.0711	2.6051
22	0.3746	0.9272	0.4040	2.6695	1.0785	2.4751
23	0.3907	0.9205	0.4245	2.5593	1.0864	2.3559
24	0.4067	0.9135	0.4452	2.4586	1.0946	2.2460
25	0.4226	0.9063	0.4663	2.3662	1.1034	2.1445
26	0.4384	0.8988	0.4877	2.2812	1.1126	2.0503
27	0.4540	0.8910	0.5095	2.2027	1.1223	1.9626
28	0.4695	0.8829	0.5317	2.1301	1.1326	1.8807
29	0.4848	0.8746	0.5543	2.0627	1.1434	1.8040
30	0.5000	0.8660	0.5774	2.0000	1.1547	1.7321
31	0.5150	0.8572	0.6009	1.9416	1.1667	1.6643
32	0.5299	0.8480	0.6249	1.8871	1.1792	1.6003
33	0.5446	0.8387	0.6494	1.8361	1.1924	1.5399
34	0.5592	0.8290	0.6745	1.7883	1.2062	1.4826
35	0.5736	0.8192	0.7002	1.7435	1.2208	1.4281
36	0.5878	0.8090	0.7265	1.7013	1.2361	1.3764
37	0.6018	0.7986	0.7536	1.6616	1.2521	1.3270
38	0.6157	0.7880	0.7813	1.6243	1.2690	1.2799
39	0.6293	0.7771	0.8098	1.5890	1.2868	1.2349
40	0.6428	0.7660	0.8391	1.5557	1.3054	1.1918
41	0.6561	0.7547	0.8693	1.5243	1.3250	1.1504
42	0.6691	0.7431	0.9004	1.4945	1.3456	1.1106
43	0.6820	0.7314	0.9325	1.4663	1.3673	1.0724
44	0.6947	0.7193	0.9657	1.4396	1.3902	1.0355
45	0.7071	0.7071	1.0000	1.4142	1.4142	1.0000

ANGLE deg	sin	cos	tan	csc	sec	cot
45	0.7071	0.7071	1.0000	1.4142	1.4142	1.0000
46	0.7193	0.6947	1.0355	1.3901	1.4396	0.9657
47	0.7314	0.6820	1.0724	1.3673	1.4663	0.9325
48	0.7431	0.6691	1.1106	1.3456	1.4945	0.9004
49	0.7547	0.6561	1.1504	1.3250	1.5243	0.8693
50	0.7660	0.6428	1.1918	1.3054	1.5557	0.8391
51	0.7771	0.6293	1.2349	1.2868	1.5890	0.8098
52	0.7880	0.6157	1.2799	1.2690	1.6243	0.7813
53	0.7986	0.6018	1.3270	1.2521	1.6616	0.7536
54	0.8090	0.5878	1.3764	1.2361	1.7013	0.7265
55	0.8192	0.5736	1.4281	1.2208	1.7435	0.7002
56	0.8290	0.5592	1.4826	1.2062	1.7883	0.6745
57	0.8387	0.5446	1.5399	1.1924	1.8361	0.6494
58	0.8480	0.5299	1.6003	1.1792	1.8871	0.6249
59	0.8572	0.5150	1.6643	1.1666	1.9416	0.6009
60	0.8660	0.5000	1.7321	1.1547	2.0000	0.5774
61	0.8746	0.4848	1.8040	1.1434	2.0627	0.5543
62	0.8829	0.4695	1.8807	1.1326	2.1301	0.5317
63	0.8910	0.4540	1.9626	1.1223	2.2027	0.5095
64	0.8988	0.4384	2.0503	1.1126	2.2812	0.4877
65	0.9063	0.4226	2.1445	1.1034	2.3662	0.4663
66	0.9135	0.4067	2.2460	1.0946	2.4586	0.4452
67	0.9205	0.3907	2.3559	1.0864	2.5593	0.4245
68	0.9272	0.3746	2.4751	1.0785	2.6695	0.4040
69	0.9336	0.3584	2.6051	1.0712	2.7904	0.3839
70	0.9397	0.3420	2.7475	1.0642	2.9238	0.3640
71	0.9455	0.3256	2.9042	1.0576	3.0716	0.3443
72	0.9511	0.3090	3.0777	1.0515	3.2361	0.3249
73	0.9563	0.2924	3.2709	1.0457	3.4203	0.3057
74	0.9613	0.2756	3.4874	1.0403	3.6280	0.2867
75	0.9659	0.2588	3.7321	1.0353	3.8637	0.2679
76	0.9703	0.2419	4.0108	1.0306	4.1336	0.2493
77	0.9744	0.2250	4.3315	1.0263	4.4454	0.2309
78	0.9781	0.2079	4.7046	1.0223	4.8097	0.2126
79	0.9816	0.1908	5.1446	1.0187	5.2408	0.1944
80	0.9848	0.1736	5.6713	1.0154	5.7588	0.1763
81	0.9877	0.1564	6.3138	1.0125	6.3925	0.1584
82	0.9903	0.1392	7.1154	1.0098	7.1853	0.1405
83	0.9925	0.1219	8.1443	1.0075	8.2055	0.1228
84	0.9945	0.1045	9.5144	1.0055	9.5668	0.1051
85	0.9962	0.0872	11.4301	1.0038	11.4737	0.0875
86	0.9976	0.0698	14.3007	1.0024	14.3356	0.0699
87	0.9986	0.0523	19.0811	1.0014	19.1073	0.0524
88	0.9994	0.0349	28.6363	1.0006	28.6537	0.0349
89	0.9998	0.0175	57.2900	1.0002	57.2987	0.0175
90	1.0000	0.0000	undefined	1.0000	undefined	0.0000

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